DOCUMENT RESUME

ED 173 111 SE 027 918

AUTHOR Haag, V. H.; And Others
TITLE Introduction to Secondary School Mathematics, Volume I (Part 2). Preliminary Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 60
NOTE 219p.; For related documents, see SE 027 917-919 and ED 159 077; Contains occasional light and broken type

EDRS PRICE MF01/PC09 Plus Postage.
DESCRIPTORS Curriculum; *Decimal Fractions; *Fractions; *Geometry; Grade 7; *Instruction; Mathematics Education; Percentage; Ratios (Mathematics); Secondary Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is part two of a three-part SMSG mathematics text for seventh-grade students. The text was written for those students whose mathematical talent is underdeveloped and is essentially the same subject matter presented in the SMSG text. Chapter topics include: (1) rational numbers and fractions; (2) non-metric geometry; (3) rational numbers and the number line; (4) decimals; and (5) ratio and percent. (MP)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED173111

SCHOOL MATHEMATICS STUDY GROUP

U S DEPARTMENT OF HEALTH
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY
SMSG

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)"

INTRODUCTION TO SECONDARY SCHOOL MATHEMATICS

VOLUME I (Part 2)

(preliminary edition)

2

SE 027 918

INTRODUCTION TO SECONDARY SCHOOL MATHEMATICS

Volume I (Part 2)

(preliminary edition)

Prepared under the supervision of a Panel consisting of:

V. H. Haag, Franklin and Marshall College

Mildred Keiffer, Cincinnati Board of Education

Oscar Schaaf, South Eugene High School, Eugene, Oregon

M. A. Sobel, Montclair State College, Upper Montclair, New Jersey

Marie Wilcox, Thomas Carr Howe High School, Indianapolis, Indiana

A. B. Willcox, Amherst College

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Copyright 1960 by Yale University.

Lithoprinted in U.S.A.
EDWARDS BROTHERS, INC.
Ann Arbor, Michigan

CONTENTS

Chapter

6. RATIONAL NUMBERS AND FRACTIONS 175
- 6-1. Division of Whole Numbers 175
- 6-2. The Invention of the Rational Numbers . . 177
- 6-3. Fractions 180
- 6-4. The Meaning of Division 183
- 6-5. Rational Numbers in General 185
- 6-6. Operations with Rational Numbers 188
- 6-7. Fractions and Multiplication 193
- 6-8. Simplifying Fractions 197
- 6-9. Multiplication of Rational Numbers by Whole Numbers 203
- 6-10. Division of Rational Numbers 207
- 6-11. Addition of Rational Numbers 214
- 6-12. Subtraction 220
- 6-13. Summary 226
- 6-14. Cumulative Review 228

7. NON-METRIC GEOMETRY 231
- 7-1. Segments 231
- 7-2. Separations 236
- 7-3. Angles and Triangles 241
- 7-4. One-to-One Correspondence 248
- 7-5. Simple Closed Curves 253
- 7-6. Summary 258
- 7-7. Chapter Review 260
- 7-8. Cumulative Review 264

8. RATIONAL NUMBERS AND THE NUMBER LINE 267
- 8-1. The Number Line 267
- 8-2. Locating Rational Numbers on the Number Line 270
- 8-3. Operations on the Number Line 273
- 8-4. Reciprocals 280
- 8-5. Comparing Rational Numbers 285
- 8-6. Mixed Numbers 293
- 8-7. Complex Fractions 302
- 8-8. Summary 304
- 8-9. Review 306
- 8-10. Cumulative Review 308

Chapter

9. DECIMALS 311
   9-1. Decimal Notation 311
   9-2. Arithmetic Operations with Decimals . . . 319
   9-3. Decimal Expansion 327
   9-4. Rounding Decimal Numerals 332
   9-5. Summary 336
   9-6. Chapter Review 337
   9-7. Cumulative Review 340

10. RATIO AND PERCENT 343
   10-1. Ratio 343
   10-2. Proportion 346
   10-3. Percent 352
   10-4. Decimals and Percent 361
   10-5. Applications of Percent 366
   10-6. Summary 381
   10-7. Chapter Review 383
   10-8. Cumulative Review 384

Chapter 6

RATIONAL NUMBERS AND FRACTIONS

6-1. Division of Whole Numbers.

Example 1: Eggs cost 60¢ a dozen. How much does one egg cost? How do we solve this problem? As we all know we divide the number of cents by the number of eggs. Our work looks like this:

$$60 \div 12 = 5.$$

The answer to the problem is that eggs cost 5 cents each.

Here is a problem which is a lot like example 1.

Example 2: Eggs cost 43¢ a dozen. How much does one egg cost? How do we solve this problem? Again we divide the number of cents by the number of eggs

$$43 \div 12.$$

Something happens now that did not happen in Example 1. This is that 43 cannot be exactly divided by 12. Other ways of saying this are:

43 is not divisible by 12;
12 is not a factor of 43;
12 is not a divisor of 43;
43 is not a multiple of 12.

What do we do about this? How much does each egg cost in Example 2? You have already worked with fractions, so you will say that each egg costs

$$\frac{43}{12}¢ \text{ or } 3\frac{7}{12}¢.$$

This is correct.

If you have never heard of fractions you would say, "You can't divide 43 by 12." You would mean that in the set of whole numbers there is no number which is equal to 43 divided by 12.

You would not know any numbers besides the whole numbers. You would be unable to divide 43 by 12.

Example 2 shows an important fact. This fact is that the set of counting numbers is not closed under division. This means that when you divide one counting number by another you do not always get an answer which is a counting number.

Sometimes when you divide one counting number by another you do get a counting number, as in Example 1. But not always. We therefore say that the set of counting numbers is not closed under division. We do not say that the set of counting numbers is sometimes closed under division and sometimes not closed under division.

Let us put it another way. We say that the set of counting numbers is closed under addition because you cannot get "outside" of the set of counting numbers by adding counting numbers. You are "closed in." You can get "outside" the set of counting numbers by dividing counting numbers. You are not "closed in."

Exercises 6-1

1. Just as we have done for multiplication and addition we shall now make a division table. In each box place the result of dividing the number on the top by the number on the left if this quotient is a whole number. If the quotient is not a whole number write "no."

	1	2	3	4	5	6	7
1	1	2	3				
2	no	1	no				
3	no	no					
4							
5							
6							
7							

2. Work the following division problems. Check your work by multiplication.

a.	12 ÷ 4	e.	54321 ÷ 3
b.	286 ÷ 11	f.	323 ÷ 17
c.	225 ÷ 15	g.	324 ÷ 18
d.	111 ÷ 37	h.	10101 ÷ 3

6-2 The Invention of the Rational Numbers.

We saw in the last section that the set of counting numbers is not closed under division. This fact does not bother us as long as the things we use mathematics to talk about occur only in whole number quantities, like marbles. But suppose we start using mathematics to talk about things like soup which may be split into parts in any way we please. Then we will find that the set of counting numbers is no longer able to satisfy all of our needs. The four examples below will make this idea clear.

Example 1: The gym teacher divides sixteen boys into two equal teams to play a game. How many boys will be on each team?

This problem is solved by dividing 16 by 2.

$$16 \div 2 = 8$$

The answer to the problem is that there are 8 boys on each team.

Example 2: The gym teacher divides seventeen boys into two equal teams to play a game. How many boys will be on each team?

Again you might start by dividing 17 by 2 as follows:

$$17 \div 2$$

Would you say that the answer is $\frac{17}{2}$ or $8\frac{1}{2}$ boys on each team? This answer would be rather unsatisfactory (especially for one of the boys). The gym teacher will not solve the problem in this way. He will do something else. Maybe he will put 8 boys on one team and nine on the other. Maybe he will make two teams of 8 boys each and use one as umpire. He will certainly not divide a boy in half.

We do not need fractions to answer questions like that in Example 2 because boys have the property of always occurring in whole number quantities. For many centuries men used numbers only to count. So they needed only the counting numbers. When men started to measure they found that they needed more numbers.

> Example 3: A woman has a ribbon 90 inches long. She wants to divide it equally among her three daughters (ages 7, 10, and 12). How long a piece of ribbon does each daughter get?

Here we divide the number of inches of ribbon by the number of daughters like this:

$$90 \div 3 = 30$$

We see that each daughter should get 30 inches of ribbon.

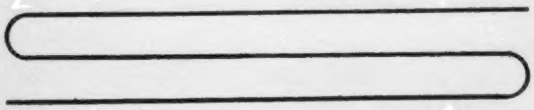
> Example 4: A woman has a ribbon 100 inches long. She wants to divide it equally among her 3 daughters. How long a piece of ribbon should each daughter get?

We start out just as we did in Example 3. We divide the number of inches of ribbon by the number of daughters like this:

$$100 \div 3 = ?$$

Since we knew something about fractions this division does not seem very hard. We say that the answer to the problem is that each daughter gets $\frac{100}{3}$ or $33\frac{1}{3}$ inches of ribbon.

But think how hard this problem must have seemed in the days when fractions had not been invented. A man who tried to solve this problem would say that there is no whole number which is equal to $100 \div 3$. He would say that 100 is not divisible by 3. On the other hand he sees that the ribbon can be folded into three pieces of equal length like this:



Then he can cut the ribbons at the folds. He now has 3 pieces of ribbon.

Each piece seems to have a definite length. Yet this length is not given by any whole number. He would like to have a number to describe this length. What is he to do? He is forced to invent a number to describe this length.

How is he to give a name to this number? The number is obtained by dividing 100 by 3. Perhaps the name should involve the numbers 100 and 3 in some way. He decides to call it by the name "$\frac{100}{3}$".

He means that $\frac{100}{3}$ is the number obtained when 100 is divided by 3. This is not the same as $\frac{3}{100}$ which is the number which would result if a woman divided 3 inches of ribbon among 100 daughters.

The numbers we get when we divide a whole number by a counting number are called rational numbers. (We never divide by zero for a very good reason that we will talk about later on.)

The word "rational" comes from the word "ratio" which means the same thing as "quotient." Rational numbers might have been called ratio-numbers or quotient-numbers.

Exercises 6-2

1. Three men who took a trip of 750 miles shared the driving equally. How far did each man drive?
2. Three men who took a trip of 700 miles shared the driving equally. How far did each man drive?
3. A bus carrying 50 passengers made a trip of 500 miles. How far did each passenger travel?

4. Perform the following division and check your work by multiplication.

Arrange your work like this:

$$\begin{array}{r} 65 \\ 42\overline{)2730} \\ \underline{252} \\ 210 \\ \underline{210} \\ 0 \end{array} \qquad\qquad \text{Check.} \quad \begin{array}{r} 65 \\ \underline{42} \\ 130 \\ \underline{260} \\ 2730 \end{array}$$

$$2730 \div 42 = 65 \qquad\qquad 42 \cdot 65 = 2730$$

a. $2752 \div 43$ d. $3293 \div 37$ g. $1729 \div 19$

b. $2915 \div 53$ e. $2093 \div 23$ h. $13431 \div 121$

c. $2916 \div 54$ f. $2541 \div 33$ i. $1001 \div 13$

6-3. Fractions.

In the last section we invented rational numbers. These numbers give us an answer to any division problem where we divide a whole number by a counting number. For example $14 \div 7 = \frac{14}{7}$. But we know that $14 \div 7 = 2$. Perhaps you will say that we can get two different answers to the problem of dividing 14 by 7. This is a good remark, but it is wrong. The fact is that

$$\frac{14}{7} = 2.$$

In other words $\frac{14}{7}$ and 2 are the same. You may say that $\frac{14}{7}$ and 2 look different. How can they be the same? The things that look different are the symbols we write down. These symbols are the names for the numbers. The symbols are not the numbers. This point was talked about in Chapter 2. Names for numbers were called numerals. The numerals

"6", "VI", "11_{five}", "6÷1"

are different names for the number 6. The numerals:

"$\frac{12}{2}$", "$\frac{18}{3}$", "$\frac{6}{1}$"

are also names for the number 6.

12

It is important to know when we are talking about a number and when we are talking about a name for a number. This is especially true in this chapter.

The numerals

$$\frac{15}{7}, \frac{6}{2}, \frac{12}{3}$$

are called fractions. In general the numeral

$$\frac{a}{b}$$

where a is a whole number and b is a counting number is called a fraction. Fractions are names for numbers. When we write

$$\frac{14}{7} = \frac{2}{1}$$

we are not saying that the fractions "$\frac{14}{7}$" and "$\frac{2}{1}$" are the same. We are saying that the numbers which these fractions stand for are the same.

Every rational number has a name which is a fraction. Some rational numbers have names which are not fractions. The number 2 is a rational number. The numeral "2" is not a fraction. Other names for 2 are $\frac{14}{7}$, $\frac{6}{3}$, $\frac{2}{1}$. All these names are fractions. It is correct to say that rational numbers are numbers which have fractions as names. The number 2 has a fraction as a name as we have just seen. Therefore 2 is rational. So are all the other whole numbers.

Exercises 6-3

1. Express the answers using fractions and without using fractions.

   a. Three men who took a trip of 750 miles divided the driving equally. How far did each man drive?

   b. A woman divides 90 inches of ribbon equally among 3 daughters. How much does each daughter get?

c. A dozen eggs cost 60 cents. How much does each egg cost?

2. Express the answers using fractions. Can you express the answers without using fractions?

a. Three men who took a trip of 700 miles divided the driving equally. How far did each man drive?

b. A woman divides 100 inches of ribbon equally among 3 daughters. How much does each daughter get?

c. A dozen eggs cost 43 cents. How much does each egg cost?

3. a. What number must you multiply 3 by to get 90?

b. What number must you multiply 12 by to get 60?

c. What number must you multiply 3 by to get 750?

d. What number must you multiply 18 by to get 54?

e. What number must you multiply 11 by to get 5280?

4. What operation did you use to find the answers in Problem 3?

5. Copy and complete the following sentences.

a. $3 \cdot$ ___ $= 90$; $90 \div 3 =$ ___.
b. $12 \cdot$ ___ $= 60$; $60 \div 12 =$ ___.
c. $3 \cdot$ ___ $= 750$; $750 \div 3 =$ ___.
d. $18 \cdot$ ___ $= 54$; $54 \div 18 =$ ___.
e. $11 \cdot$ ___ $= 5280$; $5280 \div 11 =$ ___.
f. $19 \cdot$ ___ $= 1729$; $1729 \div 19 =$ ___.

6. In each of the following problems the symbol "x" stands for a number.
In each case tell what number "x" must stand for.

a. $3 \cdot x = 15$
b. $4 \cdot x = 20$
c. $7 \cdot x = 56$
d. $18 \cdot x = 54$
e. $19 \cdot x = 1729$
f. $13 \cdot x = 1001$
g. $17 \cdot x = 17$
h. $14 \cdot x = 0$

7. What operation did you use to find the answer in Problem 6?

6-4. The Meaning of Division

We have said that division is the inverse operation of multiplication. Just what does this mean? We write

$$12 \div 3 = 4.$$

How do we check this? We write

$$3 \cdot 4 = 12.$$

These two statements say exactly the same thing in different ways. Suppose we write

$$12 \div 3 = ?$$

We might word this: what is the result of dividing 12 by 3? Another way of asking this question is:

$$3 \cdot ? = 12.$$

What number multiplied by 3 gives 12?
We have two kinds of problems using products. One kind is like this:

$$4 \cdot 5 = ?$$

which we might write

$$4 \cdot 5 = \boxed{}.$$

These are called multiplication problems.
The other kind is like this:

$$4 \cdot ? = 20$$

which we might write

$$4 \cdot \boxed{} = 20.$$

These are called division problems. To get the answer we must divide even though the division sign is not used.

In mathematics we do not use boxes or question marks. Instead we write

$$4 \cdot x = 20.$$

and ask, "What should be the value of x so that $4 \cdot x$ will be equal to 20?"

This number will be called the quotient of 20 by 4 or

$$x = 20 \div 4$$

or

$$x = \frac{20}{4}.$$

We know that this number is 5.

We see that in this way we can write division problems using only the multiplication sign.

In the same way we say that $\frac{6}{2}$ is the number x for which $2 \cdot x = 6$, $\frac{4}{3}$ is the number x for which $3 \cdot x = 4$, and $\frac{0}{3}$ is the number x for which $3 \cdot x = 0$.

Exercises 6-4

1. Copy and complete the following sentences.
   a. $\frac{6}{3}$ is the number x for which ___ $\cdot x =$ ___.
   b. $\frac{5}{2}$ is the number x for which ___ $\cdot x =$ ___.
   c. $\frac{9}{3}$ is the number x for which ___ $\cdot x =$ ___.
   d. $\frac{18}{6}$ is the number x for which ___ $\cdot x =$ ___.
   e. $\frac{10}{4}$ is the number x for which ___ $\cdot x =$ ___.
   f. $\frac{7}{1}$ is the number x for which ___ $\cdot x =$ ___.
2. In Problem 1, in which cases is the number x a whole number? Find the value for x whenever it is a whole number.

3. In each of the following sentences, find the number x which makes the statement true. (Do not change improper fractions to mixed numbers.)

   a. $2 \cdot x = 3$
   b. $5 \cdot x = 40$
   c. $4 \cdot x = 15$
   d. $7 \cdot x = 56$
   e. $3 \cdot x = 21$
   f. $5 \cdot x = 11$
   g. $10 \cdot x = 60$
   h. $8 \cdot x = 1$

4. For each of the following write a sentence which describes the problem in mathematical language. Use a letter, such as x, for the unknown number and tell, in each case, what it stands for.

   Example: Sam's father is sawing a log 12 feet long into 6 equal lengths. How long will each piece be?

   Answer: If x is the number of feet in each piece, then $6 \cdot x = 12$.

   a. If 12 cookies are divided equally among 3 boys, how many cookies should each boy receive?

   b. Mr. Carter's car used 10 gallons of gasoline for a 160 mile trip. How many miles did he drive for each gallon of gasoline used?

   c. If it takes 20 bags of cement to make a 30 foot walk, how much cement is needed for each foot of the walk?

6-5. Rational Numbers in General.

In the last section we said

$\frac{6}{2}$ is the number x for which $2 \cdot x = 6$,

$\frac{4}{3}$ is the number x for which $3 \cdot x = 4$,

$\frac{0}{3}$ is the number x for which $3 \cdot x = 0$.

From these examples can we see what $\frac{17}{9}$, $\frac{1}{5}$, $\frac{0}{42}$, $\frac{21374}{193991}$ mean? We can make a good guess at what they mean, but so far we have not said exactly what these mean. We want to tell exactly what these and all such numbers are. It is easier to do this if we use letters to stand for numbers. If we do not use letters to stand for numbers we will get a very wordy statement which will be hard to understand and impossible to remember. Here is the way we would say this. (Don't try to remember it. It is only put in as a horrible example.)

> If we write down the numerals for two counting numbers one above the other with a horizontal line between them, then the resulting symbol is a numeral for the number which when multiplied by the number whose numeral we have written on the bottom gives the number whose numeral we have written on top.

Compare this with the following definition.

Definition: If a and b are whole numbers and b is not equal to a, then

$$\frac{a}{b}$$

is the number x for which $b \cdot x = a$.

If we let a equal 6 and b equal 2 we get:

$\frac{6}{2}$ is the number x for which $2 \cdot x = 6$.

If we let a equal 4 and b equal 3 we get:

$\frac{4}{3}$ is the number x for which $3 \cdot x = 4$.

We could make the same statement without using the letter "x". We could say $\frac{a}{b}$ is the number for which $b \cdot \frac{a}{b} = a$.

Again we see that $2 \cdot \frac{6}{2} = 6$,

$$3 \cdot \frac{4}{3} = 4.$$

We should also remember that $\frac{a}{b}$ is the number we get when we divide a by b.

Exercises 6-5

1. Copy and complete.

a. $3 \cdot \underline{\quad} = 6$

b. $5 \cdot \underline{\quad} = 50$

c. $7 \cdot \underline{\quad} = 63$

d. $12 \cdot \underline{\quad} = 132$

e. $19 \cdot \underline{\quad} = 1729$

f. $35 \cdot \underline{\quad} = 1960$

2. In each case find the value of x which makes the statement correct.

a. $3 \cdot x = 6$

b. $5 \cdot x = 50$

c. $7 \cdot x = 63$

d. $12 \cdot x = 132$

e. $19 \cdot x = 1729$

f. $35 \cdot x = 1960$

3. In each case find the value of x. Express the answer as a fraction. (Do not write improper fractions as mixed numbers.)

a. $3 \cdot x = 7$

b. $5 \cdot x = 4$

c. $7 \cdot x = 5$

d. $12 \cdot x = 6$

e. $19 \cdot x = 29$

f. $35 \cdot x = 345$

4. Copy and complete.

a. $5 \cdot \frac{6}{5} = \underline{\quad}$

b. $12 \cdot \frac{7}{12} = \underline{\quad}$

c. $9 \cdot \frac{6}{9} = \underline{\quad}$

d. $3 \cdot \frac{5}{3} = \underline{\quad}$

e. $5 \cdot \frac{10}{5} = \underline{\quad}$

f. $14 \cdot \frac{13}{14} = \underline{\quad}$

5. Copy and complete.

a. $\underline{\quad} \cdot \frac{7}{5} = 7$

b. $\underline{\quad} \cdot \frac{5}{8} = 5$

c. $\underline{\quad} \cdot \frac{3}{9} = 3$

d. $\underline{\quad} \cdot \frac{14}{5} = 14$

e. $\underline{\quad} \cdot \frac{11}{9} = 11$

f. $\underline{\quad} \cdot \frac{9}{11} = 9$

6. Copy and complete.

a. $6 \cdot \frac{?}{?} = 5$

b. $12 \cdot \frac{?}{?} = 6$

c. $9 \cdot \frac{?}{?} = 7$

d. $4 \cdot \frac{?}{?} = 9$

e. $14 \cdot \frac{?}{?} = 7$

f. $11 \cdot \frac{?}{?} = 9$

g. $16 \cdot \frac{?}{?} = 1$

h. $4 \cdot \frac{?}{?} = 1$

i. $1739 \cdot \frac{?}{?} = 2403$

j. $609253 \cdot \frac{?}{?} = 17963$

7. Copy and complete.

a. $\_\_\_ \cdot \frac{6}{5} = \_\_\_$

b. $\_\_\_ \cdot \frac{9}{10} = \_\_\_$

c. $\_\_\_ \cdot \frac{3}{6} = \_\_\_$

d. $\_\_\_ \cdot \frac{24}{16} = \_\_\_$

e. $\_\_\_ \cdot \frac{1}{12} = \_\_\_$

f. $\_\_\_ \cdot \frac{0}{7} = \_\_\_$

g. $\_\_\_ \cdot \frac{5}{1} = \_\_\_$

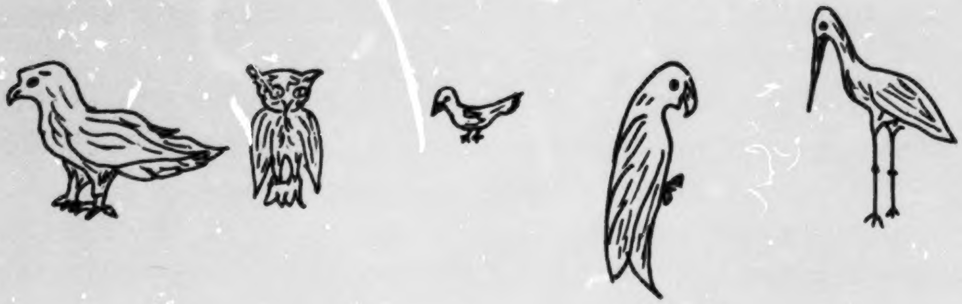
h. $\_\_\_ \cdot \frac{93147}{62973} = \_\_\_$

6-6 Operations with Rational Numbers.

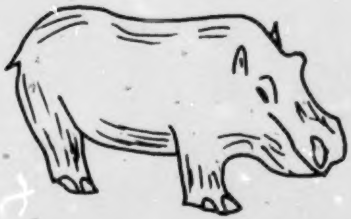
We have extended our number system to include the rational numbers. We would like to extend our operations of addition, multiplication, subtraction and division to operate on the rational numbers.

Let us think about the idea of extending the meaning of a word.

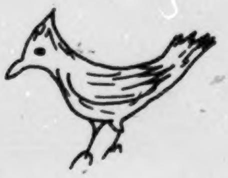
Suppose you had specimens of each of the following kinds of birds: eagle, owl, sparrow, parrot, stork. Suppose you were told that all these



creatures were birds. And suppose that you knew nothing else about birds. Now suppose that someone came to you with some other creature and asked you whether it was a bird. Suppose he came with a hippopotamus.



Would you call this creature a bird? Certainly not. Would anything prevent you from calling it a bird? No, but you wouldn't do it. Suppose someone came with a blue jay.



Would you call this creature a bird? You would, wouldn't you?

How did you make these decisions? From your limited knowledge of birds how did you decide to extend the idea of bird to include the blue jay but not to include the hippopotamus? You probably made this decision very quickly because you have been doing this sort of thing all your life. But there was a thought process involved that went like this:

The five given examples of birds are living creatures that have certain properties in common. Among these properties are:

wings, feathers, beaks, two legs, claws.

You decided to extend the use of the name "bird" to include creatures that have these properties, but not to include creatures that do not have these properties.

Now what does all this have to do with mathematics? Simply this. We are going to extend the operations of multiplication and addition to operate on the rational numbers. And we want to do this in such a way that we preserve the properties of these operations. We have already defined the operation of multiplication for certain basic products like

$$5 \cdot \frac{3}{5} = 3, \qquad 2 \cdot \frac{7}{2} = 7, \qquad 45 \cdot \frac{6}{45} = 6.$$

We used these basic products in defining the rational numbers. But how shall we define products which are not of this type, as for example:

$$\frac{6}{7} \cdot \frac{3}{5}, \qquad \frac{4}{3} \cdot \frac{3}{8}, \qquad 5 \cdot \frac{3}{4}.$$

We could define these products to be anything we like. But just as in the case of the birds, we want to extend the idea of multiplication to the rational numbers so that we preserve the properties of multiplication. The same thing is true of addition.

We saw the basic properties of these operations in Chapter 3, but we will list them again below.

Addition	Multiplication
Commutative Property $a + b = b + a$	Commutative Property $a \cdot b = b \cdot a$
Associative Property $a + (b + c) = (a + b) + c$	Associative Property $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive Property $a \cdot (b + c) = a \cdot b + a \cdot c$	
Identity Property of 0 $a + 0 = 0 + a = a$	Identity Property of 1 $a \cdot 1 = 1 \cdot a = a$
	Multiplication Property of 0 $a \cdot 0 = 0 \cdot a = 0$

We agreed that these properties were true when a, b, and c were whole numbers. Now we demand that these properties hold true when a, b, and c are rational numbers. Some examples of what we mean are:

$$\frac{2}{3} + \frac{4}{5} = \frac{4}{5} + \frac{2}{3};$$

(What property does this illustrate?)

$$\frac{2}{3} \cdot \left(\frac{4}{5} \cdot \frac{7}{9}\right) = \left(\frac{2}{3} \cdot \frac{4}{5}\right) \cdot \frac{7}{9} ;$$

(What property does this illustrate?)

$$\frac{2}{3} \cdot 1 = 1 \cdot \frac{2}{3} = \frac{2}{3} ;$$

(What property does this illustrate?)

The main purpose of this chapter is to show that the properties listed above force us to add, multiply and divide rational numbers in exactly the way we have always done.

Exercises 6-6

1. Copy and complete or find the value of x as indicated.

   a. $3 \cdot x = 4$

   b. $3 \cdot \underline{\quad} = 6$

   c. $4 \cdot \frac{?}{?} = 5$

   d. $7 \cdot \frac{6}{7} = \underline{\quad}$

   e. $\underline{\quad} \cdot \frac{5}{8} = 5$

   f. $15 \cdot x = 13$

   g. $\underline{\quad} \cdot \frac{7}{4} = \underline{\quad}$

   h. $6 \cdot x = 5$

   i. $9 \cdot \frac{?}{?} = 7$

   j. $\underline{\quad} \cdot \frac{18}{11} = \underline{\quad}$

   k. $\underline{\quad} \cdot \frac{1}{6} = \underline{\quad}$

   l. $5 \cdot \frac{?}{?} = 1$

   m. $5 \cdot x = 1$

   n. $17 \cdot \frac{?}{?} = 0$

2. Copy and complete or find the value of x as applicable.

   a. $15 \cdot x = 14$

   b. $15 \cdot \frac{?}{?} = 14$

   c. $\underline{\quad} \cdot \frac{14}{15} = 14$

   d. $15 \cdot \frac{14}{15} = \underline{\quad}$

   e. $\underline{\quad} \cdot \frac{14}{15} = \underline{\quad}$

6-7 Fractions and Multiplication

Let us try to multiply $\frac{2}{3}$ by $\frac{7}{5}$ and express the product as a fraction. We run into some difficulties when we write

$$\frac{2}{3} \cdot \frac{7}{5}$$

because we have not yet talked about products of this type. The only facts we know about the numbers $\frac{2}{3}$ and $\frac{7}{5}$ are that

$$3 \cdot \frac{2}{3} = 2 \quad \text{and} \quad 5 \cdot \frac{7}{5} = 7.$$

We will have to make use of these facts if we hope to express $\frac{2}{3} \cdot \frac{7}{5}$ as a fraction.

In order to make use of these facts we will consider a multiplication problem which is quite different from the one we started with. Instead of multiplying $\frac{2}{3}$ by $\frac{7}{5}$ we will multiply $(3 \cdot \frac{2}{3})$ by $(5 \cdot \frac{7}{5})$. In this way we will obtain a product in which $\frac{2}{3}$ and $\frac{7}{5}$ are factors. Perhaps this will lead us to a way to express $\frac{2}{3} \cdot \frac{7}{5}$ as a fraction. Look at the two multiplication problems:

$$\begin{array}{rr} & (3 \cdot \frac{2}{3}) \\ \text{times} & (5 \cdot \frac{7}{5}) \\ \hline \end{array} \qquad\qquad \begin{array}{rr} & 2 \\ \text{times} & 7 \\ \hline \end{array}$$

The factors being multiplied on the left are the same two factors as those being multiplied on the right. Why? Because

$$3 \cdot \frac{2}{3} = 2 \quad \text{and} \quad 5 \cdot \frac{7}{5} = 7.$$

Since the same numbers are being multiplied in both problems, the two products will be the same. This means that

$$(3 \cdot \tfrac{2}{3}) \cdot (5 \cdot \tfrac{7}{5}) = 2 \cdot 7.$$

Remember that $3 \cdot \frac{2}{3} = 2$ means that the symbols "$3 \cdot \frac{2}{3}$" and "2" are just different pencil scratches for the same number. The same remark applies to "$5 \cdot \frac{7}{5}$" and "7". But we do not multiply pencil scratches, we multiply numbers. So the product of two numbers does not depend on the choice of pencil scratches we use to indicate the product.

In the expression

$$(3 \cdot \tfrac{2}{3}) \cdot (5 \cdot \tfrac{7}{5}) = 2 \cdot 7$$

it will be useful to group the factors in the product on the left side in another way. The commutative and associative properties tell us that we may do this. So we re-write the product on the left as

$$(3 \cdot 5) \cdot (\tfrac{2}{3} \cdot \tfrac{7}{5}).$$

Now we have

$$(3 \cdot 5) \cdot (\tfrac{2}{3} \cdot \tfrac{7}{5}) = 2 \cdot 7,$$

or

$$15 \cdot (\tfrac{2}{3} \cdot \tfrac{7}{5}) = 14.$$

This last equality could be stated: $\frac{2}{3} \cdot \frac{7}{5}$ is the number x for which $15 \cdot x = 14$.

But we also know that $\frac{14}{15}$ is the number x for which $15 \cdot x = 14$.

Look at these two statements side by side.

$\frac{2}{3} \cdot \frac{7}{5}$	$\frac{14}{15}$
is the number x for which	is the number x for which
$15 \cdot x = 14$	$15 \cdot x = 14$

So we see that $\frac{2}{3} \cdot \frac{7}{5}$ and $\frac{14}{15}$ are the same number, or

$$\frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15} .$$

This is what we had set out to show. We have found how to express the product of $\frac{2}{3}$ and $\frac{7}{5}$ as a fraction. We have seen how the commutative and associative properties of multiplication force us to define $\frac{2}{3} \cdot \frac{7}{5}$ to be $\frac{14}{15}$.

Let us write down our steps again without explanations so we can see where we stand:

$$3 \cdot \frac{2}{3} = 2 \quad \text{and} \quad 5 \cdot \frac{7}{5} = 7;$$

$$(3 \cdot \frac{2}{3}) \cdot (5 \cdot \frac{7}{5}) = 2 \cdot 7;$$

$$(3 \cdot 5) \cdot (\frac{2}{3} \cdot \frac{7}{5}) = 2 \cdot 7;$$

$$15 \cdot (\frac{2}{3} \cdot \frac{7}{5}) = 14;$$

$$\frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15} .$$

Can you give the reason for each of the above steps? If you cannot, then read this section over again.

Let us try another multiplication problem, say $\frac{3}{4}$ times $\frac{5}{8}$. Following the steps above

$$4 \cdot \frac{3}{4} = 3 \quad \text{and} \quad 8 \cdot \frac{5}{8} = 5;$$

$$(4 \cdot \frac{3}{4}) \cdot (8 \cdot \frac{5}{8}) = 3 \cdot 5;$$

$$(4 \cdot 8) \cdot (\frac{3}{4} \cdot \frac{5}{8}) = 3 \cdot 5;$$

$$32 \cdot (\frac{3}{4} \cdot \frac{5}{8}) = 15$$

$$\frac{3}{4} \cdot \frac{5}{8} = \frac{15}{32}$$

Can you follow the steps?

In the two examples above we have shown that

$$\frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15} \quad \text{and} \quad \frac{3}{4} \cdot \frac{5}{8} = \frac{15}{32}$$

which might be written

$$\frac{2}{3} \cdot \frac{7}{5} = \frac{2 \cdot 7}{3 \cdot 5} \quad \text{and} \quad \frac{3}{4} \cdot \frac{5}{8} = \frac{3 \cdot 5}{4 \cdot 8} .$$

These are special cases of the rule:

Product Rule: if a, b, c, and d are whole numbers, with b and d not equal to zero, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

If in the Product Rule we let $a = 6$, $b = 5$, $c = 1$, $d = 2$, we find that

$$\frac{6}{5} \cdot \frac{1}{2} = \frac{6 \cdot 1}{5 \cdot 2}$$

or

$$\frac{6}{5} \cdot \frac{1}{2} = \frac{6}{10} .$$

Exercises 6-7.

1. Use the Product Rule to express the following products as fractions.

   a. $\frac{2}{7} \cdot \frac{5}{9}$

   b. $\frac{1}{3} \cdot \frac{2}{3}$

   c. $\frac{0}{5} \cdot \frac{6}{8}$

   d. $\frac{5}{4} \cdot \frac{6}{5}$

   e. $\frac{1}{6} \cdot \frac{1}{5}$

   f. $\frac{3}{4} \cdot \frac{9}{8}$

   g. $\frac{2}{5} \cdot \frac{5}{3}$

   h. $\frac{2}{5} \cdot \frac{3}{1}$

   i. $\frac{0}{3} \cdot \frac{0}{7}$

   j. $\frac{14}{19} \cdot \frac{92}{60}$

   k. $\frac{14}{60} \cdot \frac{92}{19}$

   ℓ. $\frac{x}{y} \cdot \frac{z}{w}$

2. Use the Product Rule to express the following products as fractions.

a. $\frac{3}{4} \cdot \frac{4}{3}$

b. $\frac{6}{5} \cdot \frac{5}{6}$

c. $\frac{5}{1} \cdot \frac{1}{5}$

d. $\frac{4}{7} \cdot \frac{7}{4}$

e. $\frac{2}{3} \cdot \frac{3}{2}$

f. $\frac{7}{5} \cdot \frac{5}{7}$

3. Use the Product Rule to express the following products as fractions.

a. $\frac{3}{4} \cdot \frac{5}{5}$

b. $\frac{2}{7} \cdot \frac{3}{3}$

c. $\frac{9}{2} \cdot \frac{3}{3}$

d. $\frac{8}{7} \cdot \frac{1}{1}$

4. Find fractions which make the following statements true.

a. $\frac{2}{3} \cdot \frac{?}{?} = \frac{10}{21}$

b. $\frac{4}{5} \cdot \frac{?}{?} = \frac{8}{15}$

c. $\frac{3}{4} \cdot \frac{?}{?} = \frac{9}{16}$

d. $\frac{4}{5} \cdot \frac{?}{?} = \frac{4}{5}$

e. $\frac{6}{5} \cdot \frac{?}{?} = \frac{30}{30}$

f. $\frac{1}{5} \cdot \frac{?}{?} = \frac{1}{10}$

6-8. Simplifying Fractions.

If you were asked to "simplify" the fraction

$$\frac{10}{15}$$

you would probably say that

$$\frac{10}{15} = \frac{2}{3}.$$

This is correct. But why is this correct? It turns out that this fact follows very simply from the Product Rule.

Let us see how it follows.

First of all we write

$$\frac{2}{3} \cdot 1 = \frac{2}{3} .$$

Here we have used the identity property of 1 . Now remember that when you divide a number by itself you get 1. Therefore

$$\frac{2}{2}, \frac{4}{4}, \frac{5}{5}, \frac{103}{103}$$

are other names for 1. Therefore

$$\frac{2}{3} \cdot \frac{5}{5} = \frac{2}{3} .$$

Applying the Product Rule to $\frac{2}{3} \cdot \frac{5}{5}$ we have

$$\frac{2}{3} \cdot \frac{5}{5} = \frac{10}{15} .$$

Since both $\frac{2}{3}$ and $\frac{10}{15}$ are equal to $\frac{2}{3} \cdot \frac{5}{5}$ we see that

$$\frac{10}{15} = \frac{2}{3} .$$

This is just what we wanted to show.

We will write down our steps without the explanations to that we can see what we did.

$$\frac{2}{3} \cdot 1 = \frac{2}{3}$$

$$\frac{2}{3} \cdot \frac{5}{5} = \frac{2}{3}$$

$$\frac{2}{3} \cdot \frac{5}{5} = \frac{10}{15}$$

$$\frac{10}{15} = \frac{2}{3}$$

Can you give reasons for each step?

Let us try another problem of the same kind.

Example 1: Show that $\frac{28}{21} = \frac{4}{3}$. Here is the work.

Steps	Reasons
$\frac{4}{3} \cdot 1 = \frac{4}{3}$	Identity Property of 1.
$\frac{4}{3} \cdot \frac{7}{7} = \frac{4}{3}$	"$\frac{7}{7}$" is another name for "1".
$\frac{4}{3} \cdot \frac{7}{7} = \frac{28}{21}$	Multiplication Rule.
$\frac{28}{21} = \frac{4}{3}$	$\frac{4}{3}$ and $\frac{28}{21}$ are both equal to $\frac{4}{3} \cdot \frac{7}{7}$.

It has been shown above that

$$\frac{10}{15} = \frac{2}{3} \text{ and that } \frac{28}{21} = \frac{4}{3}.$$

These are examples of the following rule.

Simplification Rule: If a, b and k are whole numbers with b and k not equal to zero then

$$\frac{a \cdot k}{b \cdot k} = \frac{a}{b}.$$

In this rule if we let $a = 2$, $b = 3$, and $k = 5$ we find that

$$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$$

or

$$\frac{10}{15} = \frac{2}{3}$$

Class Discussion Exercises 6-8a.

1. What values should we give to a, b, and k if we want the rule to tell us that $\frac{28}{21} = \frac{4}{3}$?
2. Why do we say that the fraction $\frac{4}{3}$ is simpler than the fraction $\frac{28}{21}$?

3. What values should we give to a, b and k if we want the rule to tell us that $\frac{18}{30} = \frac{3}{5}$?

4. Why do we say that the fraction $\frac{3}{5}$ is simpler than the fraction $\frac{18}{30}$?

Let us apply the rule to some problems in which the answer is not given in advance.

Example 2: Simplify the fraction $\frac{18}{42}$. First we look for a common factor of 18 and 42. Noticing that 18 and 42 are even numbers we write

$$\frac{18}{42} = \frac{9 \cdot 2}{21 \cdot 2} = \frac{9}{21} .$$

Now we notice that 3 is a factor of both 9 and 21 so that

$$\frac{9}{21} = \frac{3 \cdot 3}{7 \cdot 3} = \frac{3}{7} .$$

Therefore

$$\frac{18}{42} = \frac{3}{7}$$

We could have worked the problem in one step if we had noticed that 6 is a factor of both 18 and 42. We would then have written

$$\frac{18}{42} = \frac{3 \cdot 6}{7 \cdot 6} = \frac{3}{7} .$$

Either of these methods of arriving at the answer, $\frac{3}{7}$, is of course correct.

We often want to use the rule to do the reverse of simplification. For example we might start with the number $\frac{2}{3}$ and express this number as $\frac{10}{15}$. We can write

$$\frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15} .$$

We are still using the fact

$$\frac{a \cdot k}{b \cdot k} = \frac{a}{b}$$

but we are reading from right to left, so as to have

$$\frac{a}{b} = \frac{a \cdot k}{b \cdot k} .$$

When do we want to apply the rule in this way? Usually to express two different rational numbers as fractions having the same denominator. This is very important when we want to add, subtract, or compare rational numbers.

Example 3: Express $\frac{2}{5}$ and $\frac{3}{4}$ as fractions having the same denominator.

We solve the problem like this:

$$\frac{2}{5} = \frac{2 \cdot 4}{5 \cdot 4} = \frac{8}{20}$$

$$\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{14}{20}$$

We say that 20 is a common denominator for the fractions $\frac{2}{5}$ and $\frac{3}{4}$.

A common denominator for two fractions must be a multiple of the denominators of both given fractions.

Example 4: Express $\frac{3}{10}$ and $\frac{4}{15}$ as fractions having the same denominator.

First we look for a common multiple of 10 and 15. The most obvious common multiple of 10 and 15 is $10 \cdot 15$ or 150. We can write

$$\frac{3}{10} = \frac{3 \cdot 15}{10 \quad 15} = \frac{45}{150}$$

$$\frac{4}{15} = \frac{4 \cdot 10}{15 \cdot 10} \quad \frac{40}{150}$$

This is all right. But we could have used a smaller denominator if we had noticed that 30 is a common multiple of 10 and 15. We could then write

$$\frac{3}{10} = \frac{3 \cdot 3}{10 \cdot 3} = \frac{9}{30}$$

$$\frac{4}{15} = \frac{4 \cdot 2}{15 \cdot 2} = \frac{8}{30}$$

Any common multiple of the two denominators can be used as a common denominator. The least common multiple of the two denominators is called the least common denominator.

Exercises 6-8b

1. Show that each of the following statements is true giving steps and reasons as in Example 1.

   a. $\frac{10}{15} = \frac{2}{3}$
   b. $\frac{1}{3} = \frac{7}{21}$
   c. $\frac{5}{1} = \frac{15}{3}$
   d. $\frac{7}{4} = \frac{63}{36}$

2. Write down the statement made by the simplification rule with the following choices of a, b, and k.

   a. $a = 2$, $b = 3$, $k = 5$
   b. $a = 6$, $b = 7$, $k = 2$
   c. $a = 7$, $b = 4$, $k = 9$
   d. $a = 5$, $b = 7$, $k = 3$

3. Simplify the following fractions as in Example 2.

   a. $\frac{9}{21}$
   b. $\frac{14}{16}$
   c. $\frac{12}{24}$
   d. $\frac{15}{45}$
   e. $\frac{24}{42}$
   f. $\frac{55}{121}$
   g. $\frac{40}{60}$
   h. $\frac{75}{100}$
   i. $\frac{123}{321}$
   j. $\frac{111}{37}$
   k. $\frac{432}{234}$
   ℓ. $\frac{111111}{1111}$

4. Express each of the given rational numbers as a fraction with the indicated denominator.

a. $\frac{2}{3}$, denom. 15

b. $\frac{7}{5}$, denom. 15

c. $\frac{11}{37}$, denom. 111

d. $\frac{2}{3}$, denom. 111

e. $\frac{1}{2}$, denom. 18

f. $\frac{1}{1}$, denom. 18

g. 2, denom. 3

h. 5, denom. 3

i. $\frac{2}{5}$, denom. 15

j. $\frac{2}{5}$, denom. 35

k. $\frac{6}{15}$, denom. 35

l. $\frac{1}{3}$, denom. 15

5. Express the following groups of rational numbers as fractions with the same denominators.

a. $\frac{2}{3}$, $\frac{7}{5}$

b. $\frac{3}{10}$, $\frac{4}{15}$

c. $\frac{2}{7}$, $\frac{5}{6}$

d. $\frac{11}{37}$, $\frac{2}{3}$

e. $\frac{1}{4}$, $\frac{1}{5}$

f. $\frac{1}{2}$, $\frac{3}{4}$

g. $\frac{7}{14}$, $\frac{1}{4}$

h. $\frac{7}{30}$, $\frac{3}{20}$

i. $\frac{4}{5}$, $\frac{5}{4}$

j. $\frac{4}{1}$, $\frac{1}{4}$

k. $\frac{3}{1}$, $\frac{1}{5}$

l. 3, $\frac{1}{5}$

m. $\frac{1}{121}$, $\frac{1}{44}$

n. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$

o. $\frac{1}{6}$, $\frac{2}{15}$, $\frac{3}{10}$

6-9. Multiplication of Rational Numbers by Whole Numbers.

We know how to multiply $\frac{2}{3}$ by $\frac{7}{5}$. We have

$$\frac{2}{3} \cdot \frac{7}{5} = \frac{2 \cdot 7}{3 \cdot 5} = \frac{14}{15}$$

We know how to work with special products like $3 \cdot \frac{2}{3}$. We have

$$3 \cdot \frac{2}{3} = 2.$$

But how do we handle products like

$$5 \cdot \frac{2}{3}\ ?$$

This becomes easy if we remember that 5 is a rational number and that "$\frac{5}{1}$" is another name for 5. Therefore

$$5 \cdot \frac{2}{3} = \frac{5}{1} \cdot \frac{2}{3}$$

The multiplication problem on the right can be worked by our Product Rule

$$\frac{5}{1} \cdot \frac{2}{3} = \frac{5 \cdot 2}{1 \cdot 3} = \frac{10}{3}$$

So we see that

$$5 \cdot \frac{2}{3} = \frac{10}{3}\ .$$

In the same way we could show

> Second Product Rule: If a, b and k are whole numbers with b not equal to zero then
>
> $$k \cdot \frac{a}{b} = \frac{k \cdot a}{b}\ .$$

If in the Second Product Rule we let a equal 2, b equal 3 and k equal 5, the rule tells us that

$$5 \cdot \frac{2}{3} = \frac{5 \cdot 2}{3}.$$

Classroom Discussion Exercises 6-9a

1. Express the following products as fractions.

a. $5 \cdot \frac{2}{3}$	d. $4 \cdot \frac{3}{5}$	g. $8 \cdot \frac{1}{5}$
b. $4 \cdot \frac{2}{3}$	e. $2 \cdot \frac{1}{5}$	h. $\frac{1}{5} \cdot 8$
c. $7 \cdot \frac{2}{5}$	f. $4 \cdot \frac{1}{5}$	i. $\frac{2}{3} \cdot 4$

2. Express the following rational numbers as the product of a whole number and a fraction with numerator "1".

   (For example: $\frac{5}{3} = 5 \cdot \frac{1}{3}$)

   a. $\frac{7}{6}$ c. $\frac{3}{4}$ e. $\frac{3}{3}$

   b. $\frac{2}{5}$ d. $\frac{4}{9}$ f. $\frac{6}{9}$

Consider this word problem:

> Example 1: Eggs cost 60¢ a dozen. How much do seven eggs cost?

Try to solve this problem yourself. Did you find that seven eggs cost 35¢. How did you get this answer? You probably found the price of one egg and then multiplied this price by seven. The work looks like this:

$$60 \div 12 = 5$$

So we see that one egg costs 5 cents. Multiplying by seven:

$$7 \cdot 5 = 35$$

and seven eggs cost 35 cents.

You could also have said that the price of one egg is

$$\frac{60}{12}$$

cents and that seven eggs therefore cost

$$7 \cdot \frac{60}{12}$$

cents. We can apply our rule to this last number and write

$$7 \cdot \frac{60}{12} = \frac{7 \cdot 60}{12} = \frac{420}{12}$$

In one way of working the problem we found the answer was 35 cents. In the other way we found the answer was $\frac{420}{12}$ cents.

Do you think that these answers are different? The numerals

$$35 \text{ and } \frac{420}{12}$$

are certainly different. But remember that

$$\frac{420}{12} = 420 \div 12 = 35$$

So we see that the numbers $\frac{420}{12}$ and 35 are the same.

Here is another problem a lot like example 1.

Example 2: Eggs cost 43¢ a dozen. How much do seven eggs cost?

Here we would find that one egg costs

$$\frac{43}{12}$$

cents and seven eggs cost seven times as much or

$$7 \cdot \frac{43}{12}$$

cents. Using the Second Product Rule we would find that seven eggs cost

$$\frac{7 \cdot 43}{12} \text{ or } \frac{301}{12}$$

cents. Since 301 and 12 have no factors in common, we cannot simplify this last fraction. We do not get a whole number as the answer as we did in Example 1. So we cannot check our answer as we did in Example 1. The checking that was done in problem 1 should make you feel confident that when you get a whole number answer and a fractional answer to the same problem that the numbers will be the same.

Exercises 6-9:

1. Express the following products as fractions

   a. $5 \cdot \frac{3}{4}$
   b. $6 \cdot \frac{1}{7}$
   c. $6 \cdot \frac{2}{3}$
   d. $2 \cdot \frac{3}{4}$
   e. $\frac{4}{5} \cdot 3$
   f. $0 \cdot \frac{2}{7}$
   g. $6 \cdot \frac{5}{6}$
   h. $5 \cdot \frac{1}{5}$
   i. $\frac{0}{4} \cdot 5$

2. Express each of the following rational numbers as the products of a whole number and a fraction with numerator equal to 1.

   a. $\frac{3}{4}$
   b. $\frac{4}{5}$
   c. $\frac{7}{6}$
   d. $\frac{0}{4}$
   e. $\frac{15}{2}$
   f. 5

3. Multiply and simplify where possible

   a. $3 \cdot \frac{2}{3}$
   b. $6 \cdot \frac{2}{3}$
   c. $5 \cdot \frac{3}{10}$
   d. $6 \cdot \frac{2}{15}$
   e. $3 \cdot \frac{1}{6}$
   f. $8 \cdot \frac{1}{8}$
   g. $10 \cdot \frac{3}{25}$
   h. $7 \cdot \frac{4}{20}$
   i. $7 \cdot \frac{4}{27}$

4. A man has 7 children, 3 daughters and the rest sons. He has 20 acres of land to divide equally among all his children. What is the total number of acres that the sons will receive?

6-10 Division of Rational Numbers.

Now we want to look at the problem of dividing one rational number by another. Do you remember how we ran into trouble when we tried to divide one counting number by another? The reason for the trouble was that we cannot always find a counting number as quotient when we divide one counting number by another. That is, the set of counting numbers is not closed under division. For this reason we had to extend our number system to include the rational numbers. Do you think we shall have the same trouble when we try to divide one rational number by another? Do you think that we shall have to extend our number system again? Happily, we shall not need to extend the number system this time. The set of rational numbers (with zero excluded) is closed under division.

First, let us look at a multiplication problem. We will multiply $\frac{5}{7}$ by $\frac{14}{15}$. Using our multiplication rule we have:

$$\frac{5}{7} \cdot \frac{14}{15} = \frac{5 \cdot 14}{7 \cdot 15} \quad \text{or} \quad \frac{5}{7} \cdot \frac{14}{15} = \frac{70}{105}$$

Using the simplification rule we have:

$$\frac{70}{105} = \frac{2 \cdot 35}{3 \cdot 35}$$

and

$$\frac{2 \cdot 35}{3 \cdot 35} = \frac{2}{3}.$$

We will put all this work on one line.

$$\frac{5}{7} \cdot \frac{14}{15} = \frac{5 \cdot 14}{7 \cdot 15} \text{ and } \frac{5 \cdot 14}{7 \cdot 15} = \frac{70}{105} \text{ and } \frac{70}{105} = \frac{2 \cdot 35}{3 \cdot 35} \text{ and } \frac{2 \cdot 35}{3 \cdot 35} = \frac{2}{3}$$

Such a compound sentence is often written as

$$\frac{5}{7} \cdot \frac{14}{15} = \frac{5 \cdot 14}{7 \cdot 15} = \frac{70}{105} = \frac{2 \cdot 35}{3 \cdot 35} = \frac{2}{3}$$

Such a "chain" of equalities means that the first number is equal to the second, the second equal to the third, the third equal to the fourth and so on.

All these numbers are therefore equal and in particular the first is equal to the last or

$$\frac{5}{7} \cdot \frac{14}{15} = \frac{2}{3}.$$

These are exactly the steps you would go through if asked to multiply $\frac{5}{7}$ by $\frac{14}{15}$ and simplify. In fact you have worked this problem before. It was the last problem in the last section.

How is this related to division? Well, you remember that every division problem can be restated as a multiplication problem.

In the same way, every multiplication problem can be restated as a division problem.

The statement

$$\frac{5}{7} \cdot \frac{14}{15} = \frac{2}{3}$$

can be restated as

$$\frac{2}{3} \div \frac{5}{7} = \frac{14}{15} .$$

To make this clearer we recall that

$$b \cdot x = a$$

means that

$$x = a \div b$$

Now $\frac{14}{15}$ is the number x such that

$$\frac{5}{7} \cdot x = \frac{2}{3} .$$

Therefore $\frac{14}{15}$ is the number x such that

$$x = \frac{2}{3} \div \frac{5}{7} .$$

That is

$$\frac{14}{15} = \frac{2}{3} \div \frac{5}{7} .$$

A closer look at this last statement will give us a much clearer idea of what is going on. Since $14 = 2 \cdot 7$ and $15 = 3 \cdot 5$ we have

$$\frac{14}{15} = \frac{2 \cdot 7}{3 \cdot 5}$$

so that

$$\frac{2}{3} \div \frac{5}{7} = \frac{2 \cdot 7}{3 \cdot 5}$$

Suppose that we now replace 2, 3, 5 and 7 by a, b, c and d in order to guess the general rule. We would get

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$$

We know that this rule holds ture when a, b, c and d are 2, 3, 5 and 7, but does it always hold true? The fact is that the rule always holds true provided that we are careful never to divide by zero.

Division Rule: If a, b, c, d are whole numbers with b, c, and d not equal to zero then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$$

Let us try another example. Look at

$$\frac{4}{9} \div \frac{6}{11}$$

The rule tells us that

$$\frac{4}{9} \div \frac{6}{11} = \frac{4 \cdot 11}{9 \cdot 6}$$

Performing the indicated multiplications, the rule tells us that

$$\frac{4}{9} \div \frac{6}{11} = \frac{44}{54} .$$

Let us check to see whether this is really true.
We must check whether it is true that

$$\frac{6}{11} \cdot \frac{44}{54} \text{ is equal to } \frac{4}{9} .$$

Now

$$\begin{aligned} \frac{6}{11} \cdot \frac{44}{54} &= \frac{6 \cdot 44}{11 \cdot 54} && \text{Multiplication Rule} \\ &= \frac{6 \cdot 4 \cdot 11}{11 \cdot 6 \cdot 9} && \text{Factoring} \\ &= \frac{4 \cdot 66}{9 \cdot 66} && \text{Grouping Factors} \\ &= \frac{4}{9} && \text{Simplification Rule.} \end{aligned}$$

This shows us that

$$\frac{6}{11} \cdot \frac{44}{54} = \frac{4}{9} \quad \text{or} \quad \frac{4}{9} \div \frac{6}{11} = \frac{44}{54}$$

so that the division rule does work in this case.

How can we remember our division rule? Let us compare the division rule with the easier to remember multiplication rule.

The multiplication rule states that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

The division rule states that

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$$

The fractions "$\frac{a \cdot c}{b \cdot d}$" and "$\frac{a \cdot d}{b \cdot c}$" look somewhat alike. How are they different? The "c" and "d" have switched places. We could state the rule as follows:

To write a fraction for the quotient of two rational numbers, write a fraction which is the same as the fraction for the product of these numbers except that the numerator and denominator of the divisor fraction switch places.

If we want to write a fraction for the product

$$5 \div \frac{3}{4}$$

we first recall that

$$5 = \frac{5}{1}$$

and proceed as before. That is

$$5 \div \frac{3}{4} = \frac{5}{1} \div \frac{3}{4} = \frac{5 \cdot 4}{1 \cdot 3} = \frac{20}{3} .$$

We proceed in the same way in problems like

$$\frac{2}{3} \div 7.$$

That is

$$\frac{2}{3} \div 7 = \frac{2}{3} \div \frac{7}{1} = \frac{2 \cdot 1}{3 \cdot 7} = \frac{2}{21} .$$

Exercises 6-10

1. Perform the following divisions.

a. $\frac{2}{3} \div \frac{5}{7}$

b. $\frac{5}{7} \div \frac{2}{3}$

c. $\frac{3}{2} \div \frac{5}{7}$

d. $\frac{5}{7} \div \frac{3}{2}$

e. $\frac{2}{3} \div \frac{7}{5}$

f. $\frac{7}{5} \div \frac{2}{3}$

g. $\frac{3}{2} \div \frac{7}{5}$

h. $\frac{7}{5} \div \frac{3}{2}$

i. $\frac{2}{5} \div \frac{3}{7}$

j. $\frac{3}{7} \div \frac{2}{5}$

k. $\frac{5}{2} \div \frac{3}{7}$

l. $\frac{3}{7} \div \frac{5}{2}$

m. $\frac{2}{5} \div \frac{7}{3}$

n. $\frac{7}{3} \div \frac{2}{5}$

o. $\frac{5}{2} \div \frac{7}{3}$

p. $\frac{7}{3} \div \frac{5}{2}$

q. $\frac{2}{7} \div \frac{5}{3}$

r. $\frac{5}{3} \div \frac{2}{7}$

s. $\frac{7}{2} \div \frac{5}{3}$

t. $\frac{5}{3} \div \frac{7}{2}$

u. $\frac{2}{7} \div \frac{3}{5}$

v. $\frac{3}{5} \div \frac{2}{7}$

w. $\frac{7}{2} \div \frac{3}{5}$

x. $\frac{3}{5} \div \frac{7}{2}$

2. Divide:

a. $\frac{22}{3} \div \frac{5}{3}$

b. $\frac{4}{3} \div \frac{5}{4}$

c. $\frac{11}{3} \div \frac{7}{4}$

d. $\frac{1}{2} \div \frac{2}{3}$

e. $\frac{1}{2} \div \frac{1}{3}$

f. $\frac{2}{1} \div \frac{3}{5}$

g. $\frac{6}{7} \div \frac{5}{3}$

h. $\frac{9}{5} \div \frac{2}{1}$

i. $\frac{4}{5} \div \frac{5}{6}$

3. **Divide and simplify.**

a. $\frac{2}{3} \div \frac{5}{3}$

b. $\frac{3}{4} \div \frac{7}{4}$

c. $\frac{3}{7} \div \frac{4}{7}$

d. $\frac{5}{7} \div \frac{5}{8}$

e. $\frac{53}{3} \div \frac{53}{4}$

f. $\frac{7}{6} \div \frac{21}{10}$

g. $\frac{1}{2} \div \frac{1}{4}$

h. $\frac{1}{6} \div \frac{1}{3}$

i. $\frac{3}{4} \div \frac{3}{4}$

j. $\frac{15}{16} \div \frac{5}{4}$

k. $\frac{14}{15} \div \frac{7}{10}$

ℓ $\frac{4}{9} \div \frac{2}{3}$

m. $\frac{25}{4} \div \frac{5}{8}$

n. $\frac{14}{15} \div \frac{2}{5}$

o. $\frac{10}{11} \div \frac{12}{13}$

p. $\frac{5}{6} \div \frac{7}{8}$

q. $\frac{8}{9} \div \frac{10}{11}$

r. $\frac{1}{2} \div \frac{3}{4}$

4. **Divide and simplify when possible.**

a. $5 \div \frac{3}{4}$

b. $5 \div \frac{5}{6}$

c. $4 \div \frac{16}{3}$

d. $14 \div \frac{7}{8}$

e. $\frac{6}{5} \div 3$

f. $\frac{7}{8} \div 4$

g. $\frac{2}{3} \div 5$

h. $\frac{9}{10} \div 10$

i. $\frac{3}{7} \div 21$

5. **Divide**

a. $\frac{2}{3} \div 1$

b. $\frac{4}{9} \div 1$

c. $\frac{9}{12} \div 1$

d. $1 \div \frac{5}{7}$

e. $1 \div \frac{6}{5}$

f. $1 \div \frac{10}{11}$

g. $1 \div \frac{4}{3}$

h. $1 \div \frac{3}{4}$

i. $1 \div \frac{14}{15}$

6. **Divide and simplify.**

a. $\frac{2}{3} \div \frac{2}{3}$

b. $\frac{16}{7} \div \frac{16}{7}$

c. $\frac{29}{33} \div \frac{29}{33}$

d. $\frac{171}{654} \div \frac{171}{654}$

e. $\frac{1492}{1961} \div \frac{1492}{1961}$

f. $\frac{27803}{9959} \div \frac{27803}{9959}$

6-11. Addition of Rational Numbers.

In this section we will see how the distributive property forces us to add rational numbers as we do. We will start with the simplest kind of addition problem where the denominators are the same. Suppose we add $\frac{2}{7}$ and $\frac{3}{7}$. Now

$$\frac{2}{7} + \frac{3}{7} = 2 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} .$$

Here we have used our rule for multiplying a rational number, a whole number. Using the distributive property we obtain:

$$2 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} = (2 + 3) \cdot \frac{1}{7} .$$

The sum indicated in the parentheses is the sum of whole numbers and is easily performed:

$$(2 + 3) \cdot \frac{1}{7} = 5 \cdot \frac{1}{7} .$$

Now we may use the second Product Rule to obtain

$$5 \cdot \frac{1}{7} = \frac{5}{7} ,$$

so that

$$\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$$

Collecting our steps without explanations, we can write

$$\frac{2}{7} + \frac{3}{7} = 2 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7}$$

$$2 \cdot \frac{1}{7} + 3 \quad \frac{1}{7} = (2 + 3) \cdot \frac{1}{7}$$

$$(2 + 3) \cdot \frac{1}{7} = 5 \cdot \frac{1}{7}$$

$$5 \cdot \frac{1}{7} = \frac{5}{7} .$$

Can you give the reason for each step? The work can be written in the short way as follows:

$$\frac{2}{7} + \frac{3}{7} = 2 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} = (2 + 3) \cdot \frac{1}{7} = 5 \cdot \frac{1}{7} = \frac{5}{7}.$$

Putting letters "a", "b", "c" in place of "2", "7" and "3", in order to guess the general rule, we would have;

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b},$$

If in this formula we put in 2, 7, and 3 for a, b and c we get

$$\frac{2}{7} + \frac{3}{7} = \frac{2 + 3}{7},$$

or

$$\frac{2}{7} + \frac{3}{7} = \frac{5}{7}.$$

We can try another example. Try adding $\frac{2}{15}$ and $\frac{7}{15}$. We have

$$\frac{2}{15} + \frac{7}{15} = 2 \cdot \frac{1}{15} + 7 \cdot \frac{1}{15}$$

$$2 \cdot \frac{1}{15} + 7 \cdot \frac{1}{15} = (2 + 7) \cdot \frac{1}{15},$$

$$(2 + 7) \cdot \frac{1}{15} = 9 \cdot \frac{1}{15},$$

$$9 \cdot \frac{1}{15} = \frac{9}{15}.$$

Can you give the reasons for each step? Writing these equalities in the short way:

$$\frac{2}{15} + \frac{7}{15} = 2 \cdot \frac{1}{15} + 7 \cdot \frac{1}{15} = (2 + 7) \cdot \frac{1}{15} = 9 \cdot \frac{1}{15} = \frac{9}{15}.$$

Let us see what result we get from our formula. Here we would let

$$a = 2, \quad b = 15, \quad c = 7$$

and we would have

$$\frac{2}{15} + \frac{7}{15} = \frac{2 + 7}{15} \quad \text{or} \quad \frac{2}{15} + \frac{7}{15} = \frac{9}{15}$$

This is the correct answer. In fact the formula will always give the correct answer. We will therefore write this formula as one of our rules.

First Addition Rule: (Fractions with the same denominator). If a, b, c are whole numbers with b not equal to zero then

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} .$$

Fortunately when we wish to add numbers represented by fractions having different denominators, our method is very similar to the method used when the denominators are the same. Look at

$$\frac{2}{5} + \frac{3}{7} .$$

Where is the difficulty? The denominators are not the same. Do you remember that if we are given any two rational numbers we can express them as fractions with the same denominator? For example:

$$\frac{2}{5} = \frac{2 \cdot 7}{5 \cdot 7} = \frac{14}{35} \quad \text{and} \quad \frac{3}{7} = \frac{3 \cdot 5}{7 \cdot 5} = \frac{15}{35} .$$

Therefore

$$\frac{2}{5} + \frac{3}{7} = \frac{14}{35} + \frac{15}{35} .$$

Now the problem is reduced to that of adding rational numbers expressed as fractions with the same denominator. We have

$$\frac{14}{35} + \frac{15}{35} = \frac{14 + 15}{35} = \frac{29}{35} .$$

We can show our work again like this:

$$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7}{5 \cdot 7} + \frac{3 \cdot 5}{7 \cdot 5}$$

$$\frac{2 \cdot 7}{5 \cdot 7} + \frac{3 \cdot 5}{7 \cdot 4} = \frac{14}{35} + \frac{15}{35}$$

$$\frac{14}{35} + \frac{15}{35} = \frac{14 + 15}{35}$$

$$\frac{14 + 15}{35} = \frac{29}{35}$$

Writing our work in the short way we have:

$$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7}{5 \cdot 7} + \frac{3 \cdot 5}{7 \cdot 5} = \frac{14}{15} + \frac{15}{35} = \frac{14 + 15}{35} = \frac{29}{35}$$

Our method was to express both of the numbers $\frac{2}{5}$ and $\frac{3}{7}$ as fractions with the same denominator. Let us replace 2, 5, 3 and 7 by a, b, c and d so as to guess the general rule. We have

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{c \cdot b}{d \cdot b} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d + b \cdot c}{b \cdot d} .$$

So we would guess the rule to be:

Second Addition Rule: If a, b, c and d are whole numbers with b and d not equal to zero, then

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$$

This is the correct rule.

If in this formula we substitute 2 for a, 5 for b, 3 for c and 7 for d we get

$$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 5 \cdot 3}{5 \cdot 7} = \frac{14 + 15}{35} = \frac{29}{35}$$

This is the same result as we had above. Let us try another example, say

$$\frac{3}{10} + \frac{4}{15}$$

Using the formula we would substitute 3 for a, 10 for b, 4 for c and 15 for d. We would get

$$\frac{3}{10} + \frac{4}{15} = \frac{3 \cdot 15 + 10 \cdot 4}{10 \cdot 15} = \frac{45 + 40}{10 \cdot 15} = \frac{85}{150} .$$

We can simplify the fraction "$\frac{85}{150}$" as follows:

$$\frac{85}{150} = \frac{17 \cdot 5}{30 \cdot 5} = \frac{17}{30} .$$

Therefore

$$\frac{3}{10} + \frac{4}{15} = \frac{17}{30} .$$

We can work this problem more easily without using the rule. Look again at

$$\frac{3}{10} + \frac{4}{15} .$$

The first problem is to express the two rational numbers $\frac{3}{10}$ and $\frac{4}{15}$ as fractions with the same denominator. We look for a common multiple of 10 and 15. One such common multiple is $10 \cdot 15$. (It is clear that $b \cdot d$ is always a multiple of both b and d.) The least common multiple of 10 and 15 is 30. We therefore write

$$\frac{3}{10} = \frac{3 \cdot 3}{10 \cdot 3} = \frac{9}{30} \quad \text{and} \quad \frac{4}{15} = \frac{4 \cdot 2}{15 \cdot 2} = \frac{8}{30}$$

Therefore

$$\frac{3}{10} + \frac{4}{15} = \frac{9}{30} + \frac{8}{30} .$$

And now we can complete the problem by using the first addition rule:

$$\frac{9}{30} + \frac{18}{30} = \frac{9 + 8}{30} = \frac{17}{30} .$$

Exercises 6-11

1. Perform the indicated additions:

a. $\frac{1}{5} + \frac{3}{5}$ e. $\frac{2}{7} + \frac{6}{7}$ i. $\frac{6}{11} + \frac{7}{11}$

b. $\frac{3}{10} + \frac{4}{10}$ f. $\frac{3}{8} + \frac{6}{8}$ j. $\frac{2}{5} + \frac{1}{5}$

c. $\frac{1}{3} + \frac{1}{3}$ g. $\frac{4}{3} + \frac{7}{3}$ k. $\frac{1}{5} + \frac{1}{6}$

d. $\frac{2}{9} + \frac{5}{9}$ h. $\frac{2}{8} + \frac{3}{8}$ ℓ. $\frac{25}{87} + \frac{52}{87}$

2. Perform the indicated additions and simplify:

a. $\frac{2}{5} + \frac{3}{5}$ e. $\frac{2}{9} + \frac{4}{9}$ i. $\frac{7}{8} + \frac{5}{8}$

b. $\frac{3}{10} + \frac{5}{10}$ f. $\frac{1}{8} + \frac{5}{8}$ j. $\frac{9}{8} + \frac{11}{8}$

c. $\frac{3}{10} + \frac{5}{10}$ g. $\frac{4}{7} + \frac{3}{7}$ k. $\frac{2}{3} + \frac{4}{3}$

d. $\frac{3}{10} + \frac{7}{10}$ h. $\frac{11}{7} + \frac{10}{7}$ ℓ. $\frac{26}{87} + \frac{52}{87}$

3. Perform the indicated additions by the second addition rule and simplify where possible.

a. $\frac{2}{5} + \frac{3}{7}$ e. $\frac{5}{6} + \frac{3}{14}$ i. $\frac{3}{4} + \frac{1}{2}$

b. $\frac{2}{5} + \frac{2}{7}$ f. $\frac{4}{3} + \frac{3}{4}$ j. $\frac{1}{2} + \frac{1}{6}$

c. $\frac{1}{2} + \frac{1}{3}$ g. $\frac{4}{15} + \frac{3}{20}$ k. $\frac{2}{1} + \frac{1}{3}$

d. $\frac{3}{10} + \frac{2}{15}$ h. $\frac{1}{6} + \frac{1}{3}$ ℓ. $\frac{1}{33} + \frac{3}{22}$

4. Perform the following addition problems by finding least common denominators. Simplify where possible.

a. $\frac{3}{10} + \frac{2}{5}$
b. $\frac{2}{5} + \frac{3}{7}$
c. $\frac{5}{6} + \frac{3}{14}$
d. $\frac{4}{15} + \frac{3}{20}$
e. $\frac{4}{3} + \frac{3}{4}$
f. $\frac{1}{6} + \frac{1}{3}$
g. $\frac{3}{4} + \frac{1}{2}$
h. $\frac{1}{2} + \frac{1}{6}$
i. $\frac{1}{6} + \frac{2}{9}$
j. $\frac{1}{6} + \frac{7}{9}$
k. $\frac{4}{33} + \frac{3}{22}$
. $\frac{1}{33} + \frac{3}{22}$
m. $\frac{2}{15} + \frac{2}{25}$
n. $5 + \frac{7}{6}$
o. $4 + \frac{2}{9}$

6-12. Subtraction

Try to solve this problem.

> Example 1: Bill wants to buy a dog for 7 dollars. He has 2 dollars that he received from his Uncle Max for his thirteenth birthday. How much more money does he need?

Did you find that the answer is 5 dollars?
Did you get the answer by subtracting 2 from 7?
Here is another way of thinking of this problem.
How many dollars must Bill add to his 2 dollars in order to have 7 dollars? We could ask, what is the number x for which

$$2 + x = 7?$$

Again we see that the answer is 5. We see that we can say that

$$7 - 2$$

is the number x for which

$$2 + x = 7.$$

We see that we can state subtraction problems using only the symbol for addition.

Isn't this very similar to saying that

$$12 \div 3$$

is the number x for which

$$3 \cdot x = 12?$$

Can you see the reason for this similarity? We will put the two statements down side by side.

$7 - 2$	$12 \div 3$
is the number x	is the number x
for which	for which
$2 + x = 7$	$3 \cdot x = 12$

On the left side we have stated a subtraction problem using only the symbol for addition. On the right side we have stated a division problem using only the symbol for multiplication. Do you see the reason for the similarity yet? The reason is that subtraction is the inverse operation of addition just as division is the inverse operation of multiplication.

Put in general terms:

$$b - a$$

is the number x for which

$$a + x = b$$

Why do we bring this point up now? So that we can use what we have learned about addition to work subtraction problems.

Consider the difference

$$\frac{7}{9} - \frac{2}{9}.$$

This difference is the number x for which

$$\frac{2}{9} + x = \frac{7}{9}$$

Do you see what the number x must be? It is easier to see what x must be if we express x as a fraction with denominator 9. Then we can write

$$\frac{2}{9} + \frac{?}{9} = \frac{7}{9}$$

Now we can see that the solution is $\frac{5}{9}$. That is

$$\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$$

or

$$\frac{7}{9} - \frac{2}{9} = \frac{5}{9} .$$

This can be expressed as

$$\frac{7}{9} - \frac{2}{9} = \frac{7 - 2}{9} .$$

This problem is just a special case of the following rule:

First subtraction Rule: If a, b and c are whole numbers with b not equal to zero, then

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b} .$$

We see that this rule can be used whenever the fractions have the same denominator.

Let us try the following problem using this rule:

$$\frac{12}{25} - \frac{9}{25} .$$

We see by the rule that

$$\frac{12}{25} - \frac{9}{25} = \frac{12 - 9}{25} = \frac{3}{25} .$$

How can we check this answer? By adding $\frac{3}{25}$ to $\frac{9}{25}$ and checking whether we get $\frac{12}{25}$. Remember that

$$\frac{12}{25} - \frac{9}{25}$$

is the number x for which

$$\frac{9}{25} + x = \frac{12}{25} .$$

We see that our answer is right because

$$\frac{9}{25} + \frac{3}{25} = \frac{9+3}{25} = \frac{12}{25} .$$

Classroom Discussion Exercises 6-12a

1. Express the following numbers as fractions

a. $\frac{17}{14} - \frac{11}{14}$ d. $\frac{7}{9} - \frac{2}{9}$ g. $\frac{14}{15} - \frac{14}{15}$

b. $\frac{33}{7} - \frac{15}{7}$ e. $\frac{7}{9} - \frac{5}{9}$ h. $\frac{3}{4} - \frac{1}{4}$

c. $\frac{11}{5} - \frac{9}{5}$ f. $\frac{17}{11} - \frac{6}{11}$ i. $\frac{12}{3} - \frac{2}{3}$

And when we want to work a subtraction problem where the two numbers are expressed as fractions with different denominators, what then? If you remember how we handled this same difficulty in the addition section you will know what to do here. Consider

$$\frac{7}{8} - \frac{3}{5} .$$

Our first step is to express $\frac{7}{8}$ and $\frac{3}{5}$ as fractions having the same denominator as follows:

$$\frac{7}{8} = \frac{7 \cdot 5}{8 \cdot 5} = \frac{35}{40} \text{ and } \frac{3}{5} = \frac{3 \cdot 8}{5 \cdot 8} = \frac{24}{40} .$$

Therefore

$$\frac{7}{8} - \frac{3}{5} = \frac{35}{40} - \frac{24}{40} ,$$

and now we can use the rule to obtain

$$\frac{35}{40} - \frac{24}{40} = \frac{35 - 24}{40} = \frac{11}{40} .$$

If we want to have a subtraction formula for the difference

$$\frac{a}{b} - \frac{c}{d}$$

then we proceed as follows.

(1) Express $\frac{a}{b}$ and $\frac{c}{d}$ as fractions with the same denominators:

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} - \frac{b \cdot c}{b \cdot d}$$

(2) Next apply the preceding rule:

$$\frac{a \cdot d}{b \cdot d} - \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d - b \cdot c}{b \cdot d}$$

Now we can state the subtraction rule:

Second Subtraction Rule: If a, b, c and d are whole numbers with b and d not equal to zero, then

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - b \cdot c}{b \cdot d}$$

Let us use this rule on the last problem we worked,

$$\frac{7}{8} - \frac{3}{5},$$

to see whether the rule gives $\frac{11}{40}$ as the answer. Here we let $a = 7$, $b = 8$, $c = 3$ and $d = 4$ to obtain

$$\frac{7}{8} - \frac{3}{5} = \frac{7 \cdot 5 - 8 \cdot 3}{8 \cdot 5} = \frac{35 - 24}{40} = \frac{11}{40}.$$

Now we have two ways to subtract rational numbers. They are:

(1) Express the rational numbers as fractions with the same denominator and then use the First Rule.

(2) Use the Second Rule.

Most students find the first way to be easier than the second way.

Exercises 6-12b

1. Subtract and check by addition.

a. $\frac{9}{11} - \frac{7}{11}$

b. $\frac{53}{5} - \frac{14}{5}$

c. $\frac{3}{7} - \frac{1}{7}$

d. $\frac{4}{3} - \frac{4}{3}$

e. $\frac{17}{14} - \frac{11}{14}$

f. $\frac{17}{14} - \frac{3}{14}$

g. $\frac{3}{6} - \frac{2}{6}$

h. $\frac{3}{6} - \frac{1}{6}$

i. $\frac{8}{20} - \frac{5}{20}$

2. Subtract by expressing the numbers as fractions with the same denominator.

a. $\frac{7}{8} - \frac{3}{5}$

b. $\frac{1}{2} - \frac{1}{3}$

c. $\frac{1}{2} - \frac{1}{6}$

d. $\frac{2}{5} - \frac{1}{4}$

e. $\frac{2}{3} - \frac{4}{9}$

f. $\frac{2}{3} - \frac{2}{9}$

g. $2 - \frac{3}{2}$

h. $7 - \frac{13}{3}$

i. $\frac{14}{5} - 2$

j. $\frac{7}{9} - 1$

k. $\frac{2}{3} - \frac{3}{5}$

ℓ. $\frac{17}{7} - \frac{19}{9}$

3. Subtract by expressing the numbers as fractions with the same denominator and simplify when possible.

a. $\frac{7}{15} - \frac{3}{10}$

b. $\frac{7}{20} - \frac{1}{12}$

c. $\frac{17}{10} - \frac{5}{6}$

d. $\frac{7}{15} - \frac{2}{21}$

e. $\frac{7}{22} - \frac{3}{10}$

f. $\frac{7}{10} - \frac{3}{14}$

g. $\frac{7}{15} - \frac{1}{6}$

h. $\frac{7}{6} - \frac{5}{14}$

i. $\frac{13}{35} - \frac{3}{10}$

j. $\frac{5}{14} - \frac{4}{21}$

k. $\frac{9}{14} - \frac{12}{35}$

ℓ. $\frac{23}{35} - \frac{11}{21}$

4. Subtract using the Second Subtraction Rule and simplify when possible.

a. $\frac{6}{7} - \frac{5}{6}$

b. $\frac{17}{3} - \frac{11}{2}$

c. $\frac{15}{2} - \frac{4}{3}$

d. $\frac{11}{5} - \frac{2}{3}$

e. $\frac{5}{6} - \frac{3}{10}$

f. $\frac{11}{10} - \frac{14}{15}$

g. $\frac{5}{6} - \frac{9}{14}$

h. $\frac{1}{3} - \frac{2}{9}$

i. $\frac{7}{11} - \frac{7}{12}$

6-13 Summary

In this chapter we saw why it was necessary to extend our number system from the whole numbers to the rational numbers. This extension was necessary because the counting numbers are not closed under division. Man had need for the rational numbers when he started to use numbers to measure.

Fractions are names for rational numbers. The fractions "$\frac{4}{3}$", "$\frac{8}{6}$", "$\frac{20}{15}$" are all names for the same number.

The set of rational numbers is closed under the operations of addition, multiplication and division. We have learned certain rules for expressing in the form of a fraction the sum, difference, product and quotient of two rational numbers. These rules are not arbitrary; they follow as logical consequences of the basic properties of the operations of multiplication and addition. The basic rules of operation with rational numbers are given below.

If a, b, c and d are whole numbers with b and d not equal to 0 then:

Product Rule: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$

Second Product Rule: $c \cdot \frac{a}{b} = \frac{c \cdot a}{b}$

Simplification Rule: $\frac{a \cdot d}{b \cdot d} = \frac{a}{b}$

Division Rule: $\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$ (c not zero)

First Addition Rule: $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

Second Addition Rule: $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$

First Subtraction Rule: $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$

Second Subtraction Rule: $\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - b \cdot c}{b \cdot d}$

6-13 Chapter Review

Exercises 6-13

1. Copy and complete or find the correct value of x.

a. $2 \cdot x = 6$

b. $? \cdot \frac{4}{3} = 4$

c. $7 \cdot \frac{5}{7} = ?$

d. $? \cdot \frac{5}{4} = ?$

e. $6 \cdot \frac{?}{?} = 5$

f. $4 \cdot x = 3$

2. Multiply and simplify.

a. $\frac{1}{2} \cdot \frac{2}{3}$

b. $\frac{2}{5} \cdot \frac{5}{4}$

c. $\frac{7}{5} \cdot \frac{10}{7}$

d. $\frac{12}{3} \cdot \frac{1}{5}$

e. $\frac{2}{5} \cdot \frac{3}{7}$

f. $5 \cdot \frac{2}{9}$

g. $2 \cdot \frac{4}{7}$

h. $12 \cdot \frac{2}{3}$

i. $\frac{22}{7} \cdot \frac{31}{10}$

j. $\frac{20}{3} \cdot \frac{6}{25}$

k. $\frac{1}{3} \cdot (\frac{2}{5} \cdot \frac{1}{2})$

ℓ. $\frac{1}{2} \cdot (\frac{3}{7} \cdot \frac{14}{3})$

m. $\frac{3}{4} \cdot (\frac{2}{5} \cdot \frac{10}{3})$

n. $(\frac{4}{5} \cdot \frac{2}{7}) \cdot \frac{35}{9}$

o. $\frac{1}{2} \cdot (\frac{2}{3} \cdot \frac{3}{4})$

3. Divide and Simplify.

a. $\frac{2}{3} \div \frac{4}{3}$

b. $\frac{4}{7} \div \frac{4}{7}$

c. $\frac{4}{7} \div \frac{7}{4}$

d. $\frac{5}{9} \div \frac{2}{3}$

e. $\frac{6}{4} \div \frac{5}{2}$

f. $\frac{1}{2} \div \frac{2}{3}$

g. $\frac{2}{3} \div \frac{3}{4}$

h. $\frac{3}{4} \div \frac{4}{5}$

i. $\frac{14}{15} \div \frac{21}{10}$

j. $\frac{22}{15} \div \frac{11}{5}$

k. $\frac{9}{4} \div \frac{3}{2}$

ℓ. $\frac{1}{8} \div \frac{1}{3}$

m. $5 \div \frac{2}{7}$

n. $\frac{3}{8} \div 5$

o. $\frac{5}{9} \div 4$

p. $14 \div \frac{7}{8}$

4. Add and Simplify.

a. $\frac{1}{2} + \frac{1}{3}$

b. $\frac{1}{3} + \frac{1}{4}$

c. $\frac{7}{15} + \frac{3}{10}$

d. $\frac{5}{6} + \frac{3}{10}$

e. $\frac{2}{7} + \frac{1}{3}$

f. $\frac{3}{8} + \frac{1}{4}$

g. $(\frac{5}{12} + \frac{2}{3}) + \frac{1}{6}$

h. $(\frac{3}{2} + \frac{2}{5}) + \frac{1}{3}$

i. $(\frac{1}{2} + \frac{1}{4}) + \frac{1}{8}$

j. $\frac{2}{1} + \frac{5}{6}$

k. $2 + \frac{5}{6}$

ℓ. $\frac{3}{4} + 5$

5. Subtract and Simplify.

a. $\frac{5}{6} - \frac{1}{2}$

b. $\frac{5}{6} - \frac{1}{3}$

c. $\frac{10}{21} - \frac{5}{14}$

d. $\frac{7}{3} - 2$

e. $\frac{5}{9} - \frac{1}{3}$

f. $\frac{7}{6} - \frac{6}{7}$

g. $\frac{3}{2} - \frac{2}{3}$

h. $\frac{7}{10} - \frac{4}{15}$

i. $\frac{20}{3} - \frac{3}{4}$

j. $\frac{16}{5} - 3$

k. $5 - \frac{11}{3}$

l. $\frac{11}{9} - \frac{9}{11}$

6. Perform the indicated operations and simplify.

a. $\frac{6}{5} \cdot \frac{5}{6}$

b. $\frac{4}{15} + \frac{2}{5}$

c. $\frac{4}{15} \div 5$

d. $\frac{5}{14} + \frac{2}{7}$

e. $\frac{5}{14} \div \frac{2}{7}$

f. $\frac{2}{9} \cdot \frac{3}{4}$

g. $\frac{3}{10} \div 2$

h. $\frac{3}{10} \cdot 2$

i. $\frac{5}{6} + \frac{3}{4}$

j. $\frac{5}{6} - \frac{3}{4}$

k. $\frac{23}{14} - \frac{11}{7}$

l. $\frac{24}{25} \cdot \frac{5}{6}$

m. $\frac{2}{3} + \frac{3}{4}$

n. $\frac{14}{15} \div \frac{7}{5}$

o. $\frac{13}{7} - \frac{5}{6}$

6-14 Cumulative Review

Exercises 6-14

HOW ARE YOU DOING?

1. If N represents an even number, then N + 1 will be (a) even (b) odd (c) you cannot be sure unless you know what N is.

2. Answer the following statements true or false.

   a. When the number base is large, fewer digits are needed to represent large numbers.

   b. A multiple of 6 must be an even number.

   c. When we "carry" in an addition problem, the value of what is carried is the same in the base two as in the base ten.

d. The number 47 is a composite number.

e. In the base five system, $4 + 3 = 13_{five}$.

f. All whole numbers are counting numbers.

g. All counting numbers are whole numbers.

h. If a number is divisible by 3 and 5 it is also divisible by 15.

3. State the property expressed by the following symbols.

$$a + b = b + a$$

4. Name a whole number or numbers which will make the statement true if used in place of "a" in the following.

a. $7 + a = 5 + 7$

b. $3 \cdot a = a \cdot 3$

c. $a < 2$

5. Make a drawing for the following, if possible.

a. Two lines whose intersection is the empty set.

b. Two planes whose intersection is a point.

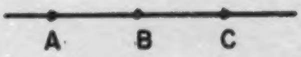
c. A point and a plane which intersect in a point.

6. Give the elements of the intersections of the following pairs of sets.

a. The days of the week and the school days of the week.

b. The odd whole numbers and the even whole numbers.

c. $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$



7. Tell which of the following are composite numbers:

a. 265 b. 99 c. 101 d. 219.

8. What is the smallest prime factor of each of the following?

a. 2,000 b. 537 c. 217 d. 10,468.

9. Write the complete factorization of 114.

19. **Perform the indicated operation and simplify.**

a. $5 \cdot \frac{2}{13}$

b. $\frac{2}{3} \cdot \frac{4}{5}$

c. $5 \div \frac{2}{3}$

d. $\frac{5}{7} + \frac{2}{3}$

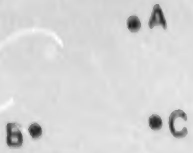
e. $\frac{2}{3} \cdot \frac{3}{5}$

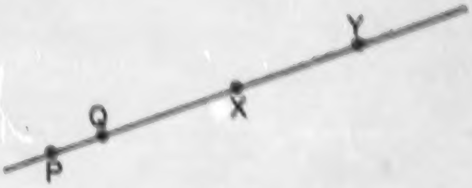
Chapter 7

NON-METRIC GEOMETRY II

7-1. Segments.

Consider three points A, B, and C as in the figure below. Do we say that any one of them is between the other two? No, we usually do not.





We use the word "between" only when the points in question are on the same line. Look at points P, Q, X, and Y above on the line PY.

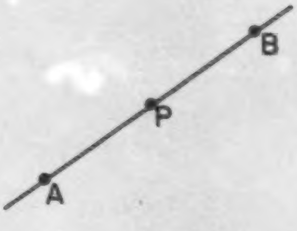
Is X between Q and Y?
Is Q between P and Y?
Is P between X and Y?

Your answers should be Yes, Yes, and No.

We know that while Q is between P and X, there are many other points between P and X. These other points have not been labeled.

When we say that a point P is between points A and B, we mean two things

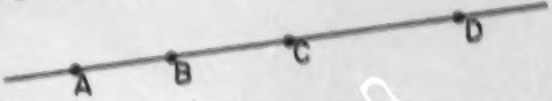
1. There is a line containing A, B, and P.
2. On that line, P is between A and B.



Look at the line AB again. Is there some point between A and P? We have not labeled any but we know that there are many there. In fact, we have the same situation for any two points in space, R and S. There are points between R and S although we may not choose to label them.

We can now say what we mean by a segment. Think of two different points L and P. The set of points consisting of L, P, and all points between L and P is called the segment LP. Points L and P are called the endpoints of the segment. We name the segment with endpoints R and S by $\overline{RS}$ or by $\overline{SR}$. A segment is a part of a line.

In the figure below, we can name segments AB, DB, and AC. Can you name other segments?



We have talked about the intersection of sets. Do you agree that in the figure above the intersection of $\overline{AC}$ and $\overline{BD}$ is $\overline{BC}$? ($\overline{AC} \cap \overline{BD} = \overline{BC}$)

Exercises 7-1a.

1. In figure 1

   a. Name two segments that intersect in C. Name the endpoints of each segment.

   b. Name three segments that intersect in point B. Name the endpoint of each segment.

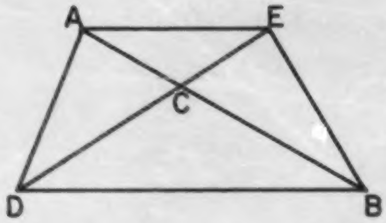


Figure 1

   c. Explain the difference between $\overleftrightarrow{AD}$ and $\overline{AD}$.

   d. What is $\overline{AB} \cap \overline{DE}$?

   e. What is $\overline{AD} \cap \overline{EB}$?

2. In figure 2

   a. Name 4 segments that intersect at Z.

   b. What is $\overline{ZY} \cap \overline{XV}$?

   c. What is $\overline{VW} \cap \overline{VZ}$?

   d. What is $\overline{XZ} \cap \overline{YV}$?

   e. What is $\overline{XV} \cap \overline{XY}$?

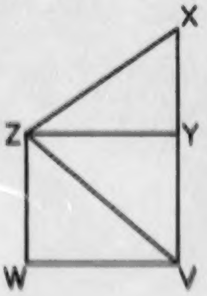


Figure 2

3. In figure 3

   a. Name 3 segments on $\overleftrightarrow{TF}$.

   b. What is $\overline{EM} \cap \overline{DM}$?

   c. How many points are there between T and F? Name two of these points.

   d. What is $\overline{TK} \cap \overline{DF}$?

   e. Name at least 6 segments on $\overleftrightarrow{AL}$.

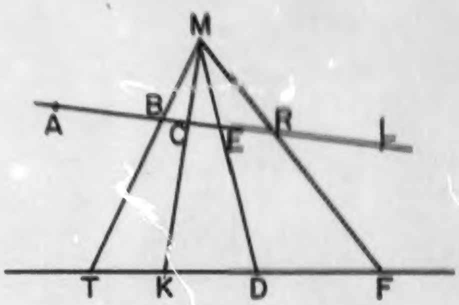


Figure 3

4. Draw a segment. Label its endpoints X and Y.

   a. Is there a pair of points of $\overline{XY}$ with Y between them?

   b. Is there a pair of points of $\overleftrightarrow{XY}$ with Y between them?

5. Draw two segments $\overline{AB}$ and $\overline{CD}$ for which $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is one point.

6. Draw two segments PQ and RS for which $\overline{PQ} \cap \overline{RS}$ is empty but $\overleftrightarrow{PQ}$ is $\overleftrightarrow{RS}$.

7. In figure 4

a. Is $\overline{AB} \cap \ell$ empty? Explain

b. Is $\overline{AC} \cap \ell$ empty? Explain

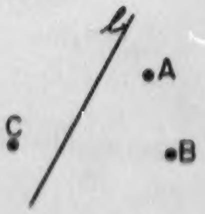


Figure 4

Union of Sets.

We have talked about the intersection of sets. It is sometimes helpful to talk about the union of sets. By the union of two sets, A and B, we mean the set that includes all the members of A and all the members of B. For example,

Let set $A = \{2, 4, 6, 8, 10\}$
Let set $B = \{5, 6, 7, 8, 9, 10\}$

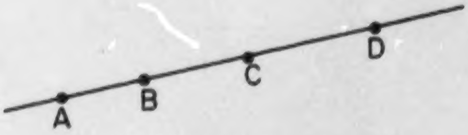
Then the union of sets A and B is

$$C = \{2, 4, 5, 6, 7, 8, 9, 10\}$$

Notice that the numbers 6, 8, and 10 appear in both sets. Each element appears just once in listing the union of two sets. The symbol used to show union is $\cup$. For example,

Let set P be the set of girls in your mathematics class.
Let set Q be the set of boys in your mathematics class.

Then $P \cup Q$ = the set of all the students in your mathematics class.



The idea of union of sets is useful in dealing with line segments. For example, in the diagram at the right, $\overline{AB} \cup \overline{BC} = \overline{AC}$. $\overline{AC} \cup \overline{BD} = \overline{AD}$.

Exercises 7-1b.

1. In each case, write the elements of the set that is the union of the given sets.

   a. R = {a, b, c, d, e}
      S = {a, e, i, o, u}

   b. P = {1, 3, 5, 7, 9, 11}
      Q = {1, 4, 9, 16}

   c. X = The whole numbers 1 through 10
      Y = The whole numbers 3 through 12

   d. A is the set of lineman on a football team
      B is the set of backs on a football team

2. Draw a line. Label three points of the line A, B, and C with B between A and C.

   a. What is $\overline{AB} \cap \overline{BC}$?

   b. What is $\overline{AC} \cap \overline{BC}$?

   c. What is $\overline{AB} \cup \overline{BC}$?

   d. What is $\overline{AB} \cup \overline{AC}$?

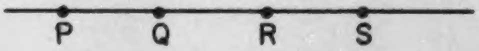
3. Let A and B be two points. Is it true that there is exactly one segment containing A and B? Draw a figure explaining this problem and your answer.

4. In the figure at the right

   a. What is $\overline{PR} \cup \overline{QR}$?

   b. What is $\overline{QR} \cup \overline{RS}$?

   c. What is $\overline{PS} \cup \overline{QS}$?



5. Draw a vertical line ℓ. Label A and B two points to the right of ℓ. Label C a point to the left of ℓ. Is $\overline{AB} \cap \ell$ empty? Is $\overline{AC} \cap \ell$ empty? Explain.

7-2. Separations.

We often find it convenient to separate sets of objects. For example, in a movie theatre we usually separate the seats on the left side from the seats on the right side by a center aisle. In baseball, we separate the points in fair territory from the points in foul territory by foul lines. In dressing, we separate the shoes from the feet by wearing socks.

This idea of separation is also important in mathematics. Three important cases of this idea will be considered below:

In figure 1, let plane ABC be the plane of the chalkboard in your classroom. This plane separates space into two sets.

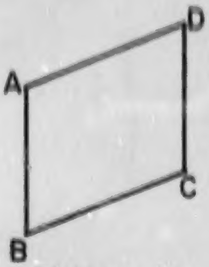


Figure 1

A. The set of points on your side of the plane of the chalkboard.

B. The set of points on the other side of the plane of the chalkboard (that is, behind the plane of the chalkboard).

The portions of space on the two sides of plane ABC are called half-spaces. The plane ABC itself, is not in either half-space.

Let R and S be any two points of space not in the plane ABC of the chalkboard. Then R and S are on the same side of the plane ABC if the intersection of RS and plane ABC is empty.

Also, R and S are on opposite sides of the plane ABC if the intersection of RS and plane ABC is not empty. In this case, there is a point of plane ABC on RS.

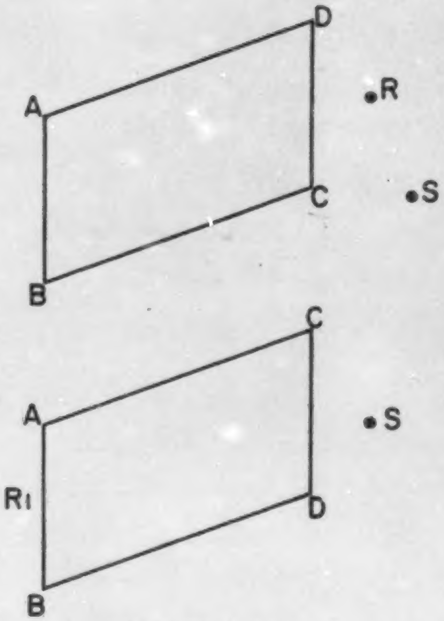


Figure 2

Thus, we have

CASE 1 - Any plane ABC separates space into two half spaces

If R and S are in the same half-space then $\overline{RS} \cap$ plane ABC is empty.

If R and S are in different half-spaces, $\overline{RS} \cap$ plane ABC is not empty.

We call plane ABC the boundary of each of the half-spaces.

Consider the plane XYZ. Do you see how the plane XYZ could be separated into two half-planes? What would be the boundary of the two half-planes?

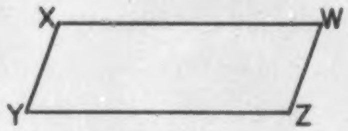


Figure 3

Do you agree that a line such as ℓ, would separate XYZ into two half-planes? We call line ℓ the boundary of the two half-planes.

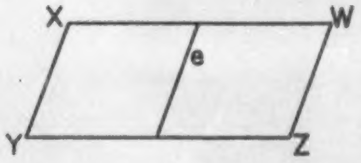
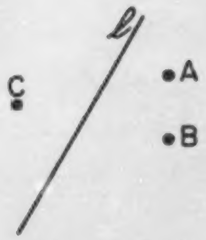


Figure 4



Look at figure 5. It consists of the line ℓ and three points A, B, and C.

Is $\overline{AB} \cap \ell$ the empty set?
Is $\overline{BC} \cap \ell$ the empty set?
Is $\overline{AC} \cap \ell$ the empty set?

If your answers were yes, no, no, you were correct. Thus, we have

CASE 2 - Any line ℓ of plane XYZ separates the plane into two half-planes.

Recall that planes and lines extend without limit. Therefore each half-plane extends without limit.

We call the two half-planes into which a line separates a plane, the sides of the line. For example, in figure 6, the line ℓ separates the plane ABC into two half-planes. The two half-planes are called the sides of ℓ. We name the sides of ℓ by saying the A-side of ℓ or the C-side of ℓ. In figure 6, the B-side of ℓ = the A-side of ℓ.

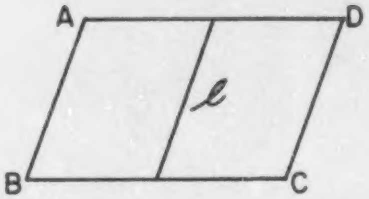


Figure 6

Now, consider the line ℓ. What do we mean by a half-line? Do you agree that a point P separates line ℓ into two half-lines? We call point P the boundary of the two half-lines.

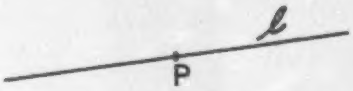


Figure 7

CASE 3 - Any point P of line ℓ separates the line ℓ into two half-lines.

In figure 8, A and C are on the same half-lines. A and B are on different half-lines. Can you see that, if two points are on different half-lines of the same line then the boundary must lie between them?

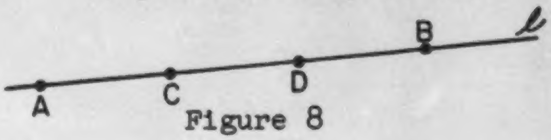


Figure 8

Exercises 7-2a.

1. Class discussion exercise - Explain how each of the following may be used as an example of a separation
   - a. The plane of the net on a tennis court
   - b. First base on a baseball field
   - c. A movie screen
   - d. The right edge of your desk
   - e. A window in your living room
   - f. The 8-inch mark on a 12-inch ruler
   - g. The goal line on a football field

2. In figure 9, are the following statements true or false? Line ℓ and points P, Q, R and S are in one plane. Give reasons for your answer.

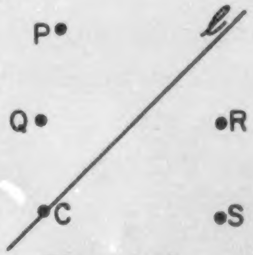


Figure 9

   - a. The R-side of ℓ is the same as the S-side of ℓ.
   - b. The S-side of ℓ is the same as the Q-side of ℓ.
   - c. $\ell \cap \overline{PQ}$ is empty
   - d. $\ell \cap \overline{RS}$ is empty
   - e. $\ell \cap \overline{QS}$ is empty
   - f. $\ell \cap \overline{PR}$ is empty

3. In figure 10,

   a. Does $\overleftrightarrow{PQ}$ separate the plane ADC?

   b. Does $\overline{PQ}$ separate the plane ADC?

   Explain your answers.

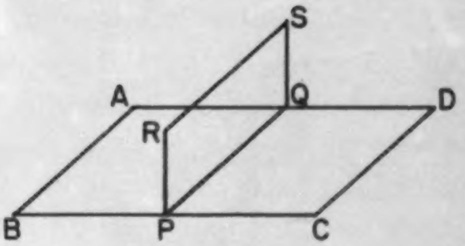


Figure 10

4. Draw two horizontal lines k and m on your paper. These lines are parallel. Label point P on line m. Is every point on m on the P-side of k? Explain.

5. The idea of a plane separating space is similar to the idea of the surface of a box separating the inside from the outside. If P is a point on the inside and Q a point on the outside of a box, does $\overline{PQ}$ intersect the surface?

*6. Explain how the union of two half-planes might be a plane.

*7. If A and B are points on the same side of plane RST (in space) must $\overleftrightarrow{AB} \cap$ plane RST be empty? Can $\overleftrightarrow{AB} \cap$ plane RST be empty?

Another useful idea in geometry is that of the ray. A ray is a half-line together with its endpoint. A ray has one endpoint. Thus, a ray without its endpoint is a half-line. We usually draw a ray like this, ⟶ . If A is the endpoint of a ray and B is another point of the ray we denote the ray by $\overrightarrow{AB}$. Note that $\overrightarrow{BA}$ is not the same as $\overrightarrow{AB}$. We use the term ray in the same sense in which it is used in "ray of light".

In everyday language, we sometimes do not use the words lines, rays, and segments as words with different meanings. In geometry, we should be careful to use them precisely. A "line of sight" really is a ray because it has an endpoint. Thus, you do not describe somebody as in your line of sight if he is behind you.

The right field foul line in baseball refers to a segment and a ray. The segment extends from home plate through first base to the bottom of the ball park fence. It stops at the fence. The ray starts on the ground and goes up the fence. What happens to a home run ball after it leaves the park makes no difference to the play in a major league game.

Exercises 7-2b.

1. Use figure 11 to explain the meanings of
   a. $\overleftrightarrow{RS}$
   b. $\overline{RS}$
   c. $\overrightarrow{RS}$

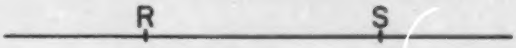


Figure 11

2. Using figure 12
   a. What is $\overline{PL} \cup \overline{LK}$?
   b. What is $\overline{PL} \cap \overline{LK}$?
   c. What is $\overrightarrow{LP} \cup \overrightarrow{LK}$?

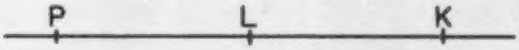


Figure 12

3. Draw a line containing points A, B, and C in order from left to right. What is $\overrightarrow{AB} \cap \overrightarrow{BA}$? What is $\overrightarrow{BA} \cap \overrightarrow{CB}$? $\overrightarrow{AC} \cap \overrightarrow{BA}$?
4. Draw a horizontal line. Label four points of it A, B, C, and D in that order from left to right. Name two rays (using pairs of these points to name them).

7-3. Angles and Triangles.

Angles.

Some of the most important ideas of geometry deal with angles and triangles. An angle is a set of points consisting of two rays with an endpoint in common and not both in the same straight line.

For example, in figure 1, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are two rays. B is the common endpoint and A, B, and C are not on the same line. Thus, the set of points consisting of all the points of BA together with all the points of BC is called the angle ABC.

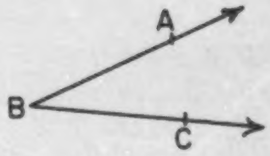


Figure 1

An angle may be considered to be the union of two rays. The common point B is called the vertex of the angle. The rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are called the rays (or sometimes the sides) of the angle. An angle has exactly one vertex and exactly two rays.

Consider the angle drawn in figure 2. Remember that R merely represents a point and that $\overrightarrow{RS}$ and $\overrightarrow{RT}$ merely represent rays. Similarly, figure 2 represents an angle.

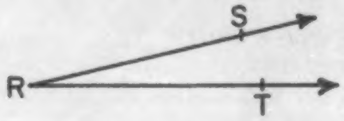


Figure 2

It is often necessary to name angles. The angle in figure 2 is named $\angle$ SRT or $\angle$ TRS. The letter at the vertex is always listed in the middle.

Class Exercise

1. Draw an angle ABC.

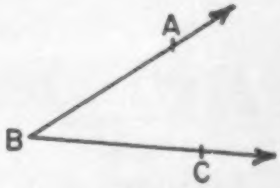


Figure 3

2. You can see that the angle separates the plane.

3. Shade the portion of the plane that appears outside the angle, as shown. We call this shaded portion the exterior of the angle.

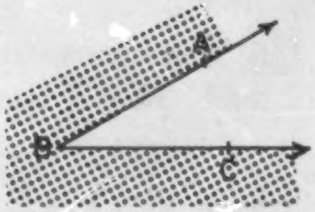


Figure 4

4. Shade the portion of the plane that appears inside the angle, as shown. We call this shaded portion the interior of the angle.

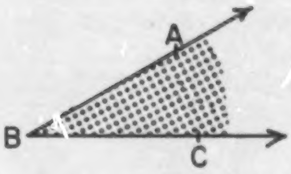


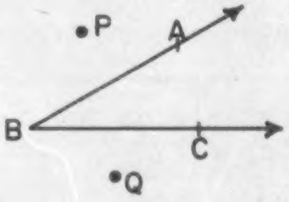
Figure 5

5. Explain the following statement.

   "The interior of $\angle$ ABC can be defined as the intersection of the A-side of the line BC and the C-side of the line AB."

Note that the interior of an angle is the intersection of two half-planes. The interior of an angle does not contain the angle. The exterior of an angle is the set of all points of the plane not on the angle or in the interior.

6. Another way to indicate the exterior of $\angle$ ABC is to say that it is the union of the P-side of $\overleftrightarrow{AB}$ and the Q-side of $\overleftrightarrow{BC}$.



Exercieses 7-3a.

1. In figure 6,

   a. Name the angle in two different ways.

   b. Name the vertex of the angle.

   c. Name the rays (or sides) of the angle.

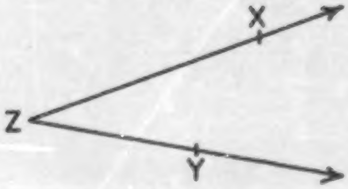


Figure 6

2. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.

   a. Shade the C-side of $\overleftrightarrow{AB}$.

   b. Shade the A-side of $\overleftrightarrow{BC}$.

   c. What set is now doubly shaded?

3. In figure 7, what is

   a. $\overrightarrow{VT} \cup \overrightarrow{VW}$?

   b. $\overleftrightarrow{TV} \cap \overleftrightarrow{WV}$?

   c. $\overrightarrow{VT} \cap \overrightarrow{VW}$?

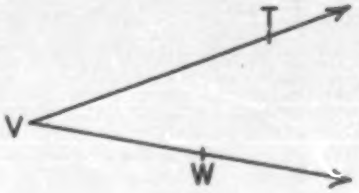


Figure 7

Triangles.

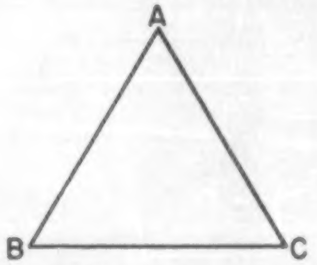


Figure 8

Let A, B, and C be three points not all on the same straight line. The triangle ABC, written as ΔABC, is the union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. You will recall that the union of two sets consists of all of the elements of the one set together with all the elements of the other set. In this case, we have the union of three sets. The union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ consists of all the elements of the three sets taken together.

We may define the $\triangle ABC$ in another way. The triangle ABC is the set of points consisting of A, B, and C, and all points of $\overleftrightarrow{AB}$ between A and B, all points of $\overleftrightarrow{AC}$ between A and C, and all points of $\overleftrightarrow{BC}$ between B and C. The points A, B, and C are the *vertices* of $\triangle ABC$. We say "vertices" when we talk about more than one vertex.

Angles of a Triangle.

We speak of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ as the *sides* of the triangle. We speak of $\angle ABC$, $\angle ACB$, and $\angle BAC$ as the *angles* of the triangle. These are the angles determined by the triangle.

Do you see that the sides of a triangle are contained in the triangle? This is true because the sides of a triangle are segments. Are the angles of a triangle contained in the triangle? The answer is no because an angle is the union of two rays with an endpoint in common. Each ray has only one endpoint and extends beyond the triangle. You may wonder why we call $\angle ABC$ an angle of $\triangle ABC$ when $\angle ABC$ is not contained in $\triangle ABC$. But think about it. We speak of the graduates *of* a school even though graduates are not *in* the school.

Note that a triangle is a set of points in exactly one plane. Every point of the triangle RST is in the plane RST. Would you say that $\triangle RST$ separates the plane in which it lies? It is true that it does. The $\triangle RST$ has an interior and an exterior. The interior is the intersection of the interiors of the three angles of the triangle. The exterior is the set of all points of plane RST not on $\triangle RST$ or in the interior of $\triangle RST$.

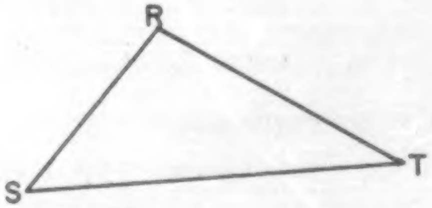


Figure 9

Exercises 7-3b.

1. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.

   a. Shade the C-side of $\overleftrightarrow{AB}$. Shade the A-side of $\overleftrightarrow{BC}$. What set is now doubly shaded?

   b. Shade the B-side of $\overleftrightarrow{AC}$. What set is now triply shaded?

2. Label three points X, Y, and Z not all on the same line.

   a. Draw $\angle XYZ$ and $\angle XZY$. Are they different angles? Why?

   b. Draw $\angle YXZ$. Is this angle different from both of the other two you have drawn?

   c. Each angle is a set of points in exactly one plane. Why is this true?

3. Draw a triangle ABC

   a. In the triangle, what is $\overline{AB} \cap \overline{AC}$?

   b. Does the triangle contain any rays or half-lines? Why?

   c. In the drawing extend $\overline{AB}$ in both directions to obtain $\overleftrightarrow{AB}$. What is $\overline{AB} \cap \overleftrightarrow{AB}$?

   d. What is $\overleftrightarrow{AB} \cap \triangle ABC$?

   e. What is $\overrightarrow{AB} \cup \overrightarrow{AC}$?

4. In figure 10

   a. What is $\overline{YW} \cap \triangle ABC$?

   b. Name the four triangles in the figure.

   c. Which of the labeled points, if any, are in the interior of any of the triangles?

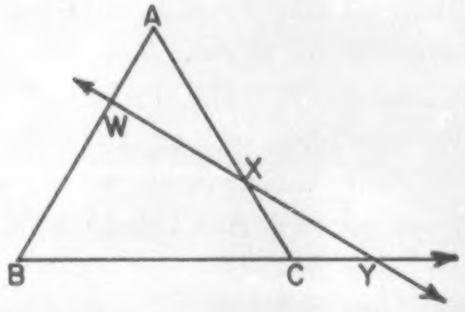


Figure 10

d. Which of the labeled points, if any, are in the exterior of any of the triangles?

e. Name a point on the same side of $\overleftrightarrow{WY}$ as C and one on the opposite side.

5. On your paper, make a copy of figure 10.

a. Label a point P not in the interior of any of the triangles.

b. Label a point Q inside two of the triangles.

c. If possible, label a point R in the interior of ΔABC, but not in the interior of any other of the triangles.

6. If possible, make sketches in which the intersection of a line and a triangle is:

a. The empty set

b. A set of two elements

c. A set of one element

d. A set of exactly three elements

7. If possible, make sketches in which the intersection of two triangles is:

a. The empty set

b. Exactly two points

c. Exactly four points

d. Exactly five points

8. In figure 11, what are the following:
   a. $\angle ABC \cap \overleftrightarrow{AC}$
   b. $\triangle ABC \cap \overleftrightarrow{AB}$
   c. $\ell_1 \cap \angle ACB$
   d. $\overline{AB} \cap \ell_2$
   e. $\overline{BC} \cap \triangle ABC$
   f. $\overrightarrow{BC} \cap \angle ACB$
   g. $\angle ABC \cap \triangle ABC$
   h. $\overrightarrow{AC} \cup \overrightarrow{CB}$

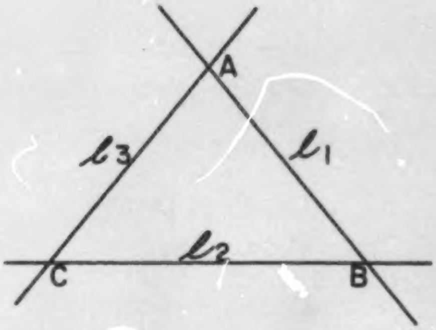


Figure 11

*9. In a plane if two triangles have a common side must their interiors intersect? If three triangles have a common side, must some two of their interiors intersect? Make drawings to explain your answers?

10. BRAINBUSTER: Draw $\angle ABC$. Label points X and Y in the interior and P and Q in the exterior.
   a. Must every point of $\overline{XY}$ be in the interior?
   b. Is every point of $\overline{PQ}$ in the exterior?
   c. Can you find points R and S in the exterior so that $\overline{RS} \cap \angle ABC$ is empty?
   d. Can $\overline{XP} \cap \angle ABC$ be empty?

7-4. One-To-One Correspondence.

In the first part of Chapter III we talked about one-to-one correspondence. We learned that early man kept track of the number of sheep in his flock as follows. For each sheep he put a stone in a pile. In order to make sure that no sheep were missing he took a stone out of the pile as each sheep went into the pen. If there were no stones left in the pile when the last sheep was in the pen he knew that no sheep were missing. This process of matching each member of one set with a member of another set is an example of one-to-one correspondence.

Let us consider other examples of one-to-one correspondence, or matching. When Mrs. Barber served tea she put a set of saucers on the table, one for each guest. Susan Barber then placed a cup on each saucer. Thus, there was a one-to-one correspondence between the set of saucers and the set of cups on the table.

At the start of a major league baseball game nine players ran onto the field and took their positions. There was a one-to-one correspondence between the set of players and the set of positions.

Let U be the set of human noses in your classroom.
Let V be the set of humans in your classroom.

Is there a one-to-one correspondence between U and V? For each nose is there a matching human? For each human is there a matching nose?

Is showing that two sets X and set Y are in one-to-one correspondence we must show that

a. For each element in X there is a corresponding element in Y.

and

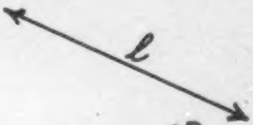
b. For each element in Y there is a corresponding element in X.

Let us see how a one-to-one correspondence is used in geometry.

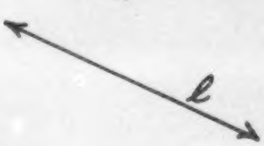
Follow the directions. Answer the questions as you go along.

Class Exercise

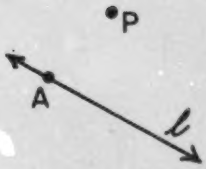
1. Draw a line and label it ℓ.



2. Locate a point above line ℓ and label it P.



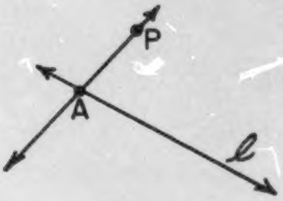
3. Locate a point on line ℓ and label it A.



4. Draw line $\overleftrightarrow{PA}$

   a. Does $\overleftrightarrow{PA}$ intersect line ℓ ?

   b. How many elements are there in the intersection set of $\overleftrightarrow{PA}$ and ℓ ?

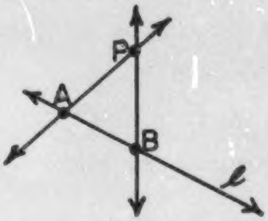


5. Locate a second point on line ℓ and label it B.

6. Draw line $\overleftrightarrow{PB}$.

   a. Does $\overleftrightarrow{PB}$ intersect ℓ ?

   b. How many elements are in the intersection set of $\overleftrightarrow{PB}$ and ℓ ?

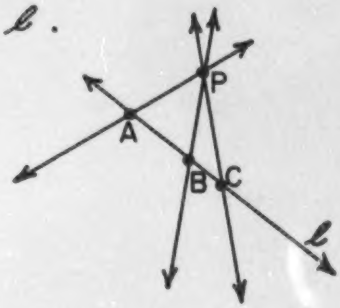


7. Locate a third point C on line ℓ .

8. Draw line $\overleftrightarrow{PC}$.

   a. Does $\overleftrightarrow{PC}$ intersect ℓ ?

   b. How many elements are in the intersection set of $\overleftrightarrow{PC}$ and ℓ ?



9. a. Is it possible to locate more points on ℓ ?

   b. Through each additional point marked on ℓ can you draw a line that also passes through point P?

10. Let all lines which intersect ℓ and pass through P be elements of a set called K.

    a. How many elements of K have been drawn so far?

    b. How many points have been located on line ℓ so far?

    c. Is there a point on ℓ for each element of K drawn so far?

    d. Is there an element of K for each point located on ℓ ?

    e. Can you match "one-to-one" the elements of K and the points located so far?

    f. Do you think that if more points were located on ℓ and more elements of K were drawn that a one-to-one correspondence between the sets could be shown?

    g. Supply the missing words:
    To a __________ through P and intersecting ℓ
    there corresponds a _________ on ℓ and to a
    __________ on ℓ there corresponds a ________
    through P and intersecting ℓ.

Do you agree that there is a one-to-one correspondence between the set K of lines and the set ℓ of points?

The study of numbers and the study of geometry is combined in a branch of mathematics called "coordinate geometry." Great progress in science and technology have been made by the use of this combination. To understand how this combination of number and geometry is made to work we will use the ideas of "line in geometry", "number line", and "one-to-one correspondence".

Exercises 7-4

1. Each desk in a classroom is assigned to a pupil.

   a. When all pupils are present, is there a one-to-one correspondence between the pupils and desks in the classroom?

   b. Is there always a one-to-one correspondence between the pupils and desks in a classroom? Explain your answer.

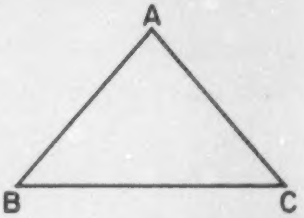
2. If you have a full set of fingers (including thumbs)

   a. Is there a one-to-one correspondence between the fingers on your left hand and the fingers on your right hand?

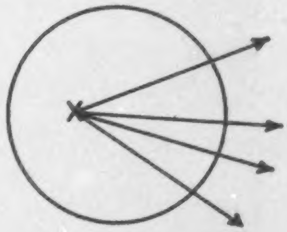
   b. Copy and complete: For each finger on my left hand there is a ________ finger on my ________ ________.

3. Two basketball teams are playing on a court. Show that there is a one-to-one correspondence between the members of one team and the members of the other team.

4. In the triangle ABC show that there is a one-to-one correspondence between points A, B, and C and the sides opposite these points.



5. Draw a circle as shown at the right. Label a point X in the interior of the circle. Let R be the set of all rays having X as an endpoint. We understand that the elements of R are in the same plane as the circle. Draw several rays of set R. Copy and complete the following



   a. For every ______ of the circle there is ______ ray of set R cutting the circle.

b. For every __________ of set R, there is __________ __________ in which the ray cuts the circle.

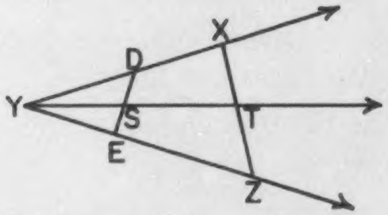
c. Is there a one-to-one correspondence between set R and the circle?

6. In the diagram at the right, XYZ is an angle. $\overrightarrow{YX}$, $\overrightarrow{YT}$, and $\overrightarrow{YZ}$ are elements of set K. Set K is the set of all the rays through Y which do not contain points in the exterior of $\angle XYZ$.

$\overline{DE}$ and $\overline{XZ}$ are segments joining points on $\overline{YZ}$ with points on $\overline{YX}$.

a. Show that there is a one-to-one correspondence between set K and $\overline{DE}$.

b. Show that there is a one-to-one correspondence between set K and $\overline{XZ}$.



c. Show that there is a one-to-one correspondence between $\overline{DE}$ and $\overline{XZ}$.

*7. Show that there is a one-to-one correspondence between the set of even whole numbers and the set of odd whole numbers.

*8. Show that there is a one-to-one correspondence between the set of even whole numbers and the set of whole numbers.

7-5. Simple Closed Curves.

In newspapers and magazines you often see graphs like those in figures 1 and 2.

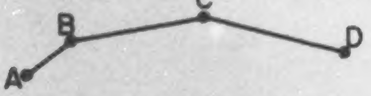


Figure 1

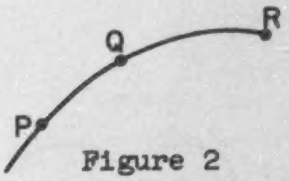


Figure 2

These graphs represent curves. We shall consider curves to be special types of sets of points. Sometimes, paths that wander around in space are thought of as curves. But in this section we shall consider only curves that are in one plane. Such curves may be drawn on a chalkboard or on a sheet of paper.

A curve is a set of points which can be represented by a pencil drawing without lifting the pencil off the paper. Segments and triangles are examples of curves we have already studied. Curves may or may not contain portions that are straight.

One important type of curve is called a broken-line curve. It is the union of several line segments. That is, it consists of all the points on several line segments. Figure 1 represents a broken line curve. A, B, C, and D are marked as points on the curve. We also say that the curve contains or passes through these points. Figure 2 also represents a curve. In figure 2 points P, Q, and R are marked on the curve. Of course, we think of the curve as containing many points other than P, Q, and R.

A curve is said to be a simple closed curve if

a. the drawing starts and stops at the same point.
b. no other point is touched twice by the pencil mark.

Figures 3, 4, 5, and 6 are examples of simple closed curves.

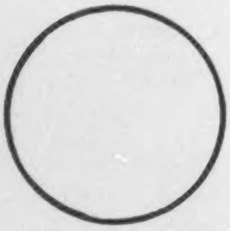


Figure 3

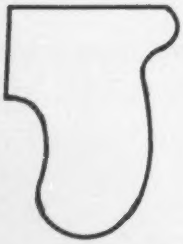


Figure 4

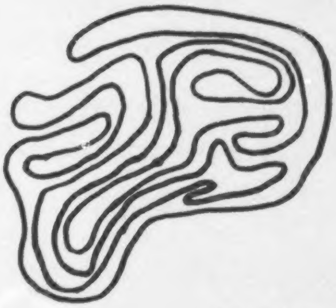


Figure 5

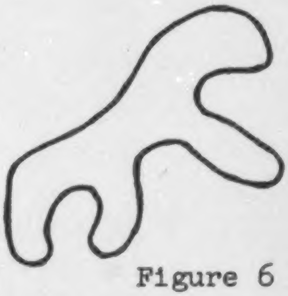


Figure 6

Figures 3a, 4a, 5a, and 6a are examples of curves which are not simple closed curves.

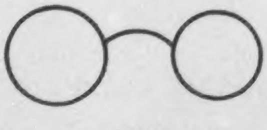


Figure 3a

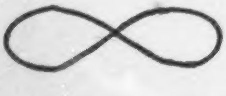


Figure 4a

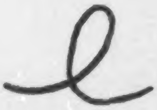


Figure 5a

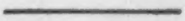


Figure 6a

Figure 7 represents two simple closed curves.

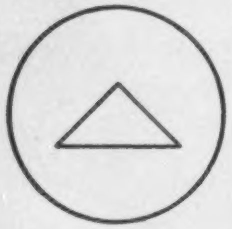


Figure 7

The boundary of a state like Iowa or Utah on an ordinary map represents a simple closed curve. A fence which extends all the way around a cornfield or a city park suggests a simple closed curve.

You can see that a simple closed curve has an interior and an exterior. Also, any line or curve containing a point in the interior and a point in the exterior must intersect the closed curve.

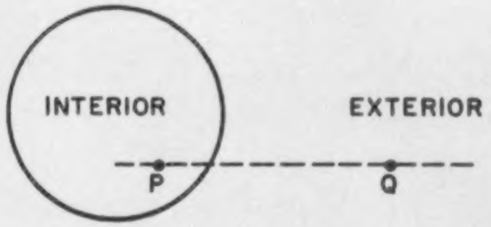


Figure 8

In figure 8, P is in the interior of the closed curve and Q is in the exterior of the closed curve. The dotted line which contains P and Q must cut the curve.

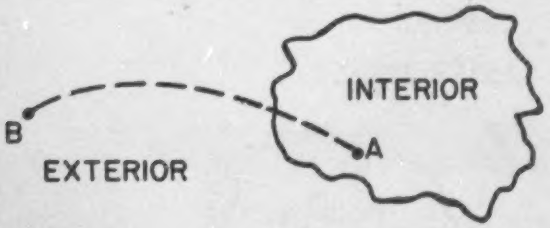


Figure 9

In figure 9, A is in the interior of the closed curve and B is in the exterior of the closed curve. The dotted curve which contains A and B must cut the curve.

Any two points in the interior or any two points in the exterior of a simple closed curve may be joined by a broken-line curve which does not intersect the simple closed curve.

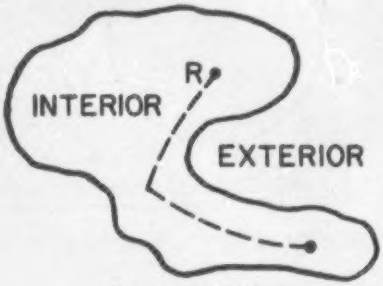


Figure 10

In figure 10, R and S are points in the interior of a simple closed curve. R and S are joined by a broken line curve (dotted line) which does not intersect the simple closed curve.

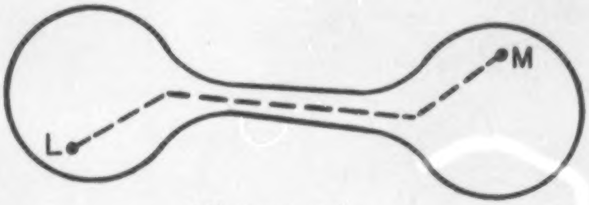
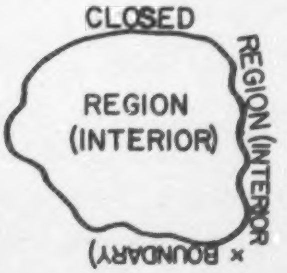


Figure 11

In figure 11, L and M are points in the interior of a simple closed curve. L and M are joined by a broken-line curve (dotted line) which does not intersect the simple closed curve.

We call the interior of a simple closed curve a region.

We call the interior of a simple closed curve together with its boundary a closed region.



Exercises 7-5.

1. Draw the simple closed curves described below:

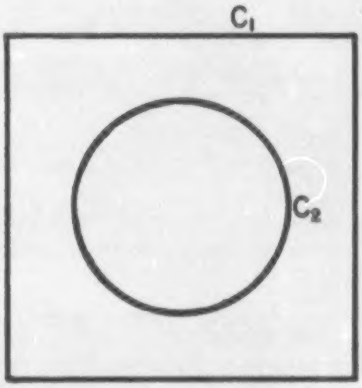
   a. one which is the union of four segments.

   b. one which is the union of five segments.

   c. one which is the union of three segments.

2. Draw a figure representing two simple closed curves whose intersection is exactly two points.

3. In the figure at the right, the two simple closed curves are represented as C_1 and C_2.



   a. Make a copy of this figure.

   b. Shade the exterior of C_2 with horizontal lines.

   c. Shade the interior of C_1 with vertical lines.

   d. Using the word "intersection" describe in words the portion of the plane that is doubly shaded.

   e. Describe the portion of the plane that is singly shaded.

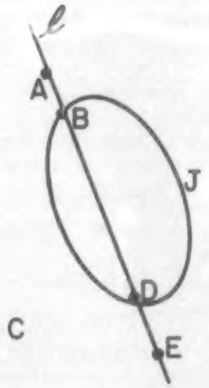
4. Look at a map of the United States.

   a. Does the boundary of Colorado represent a simple closed curve?

   b. Does the boundary of Arizona represent a simple closed curve?

   c. Does the union of the boundaries of Colorado and Arizona represent a simple closed curve? Why?

5. The line ℓ and the simple closed curve J are shown in the figure at the right.



   a. What is $J \cap \ell$?

   b. Draw a similar figure and shade the intersection of the interior of J and the C-side of ℓ.

   c. Describe in terms of rays the set of points on ℓ not in the interior of J.

6. BRAINBUSTER. Draw two simple closed curves whose interiors intersect in three different regions.

7-6. Summary.

Many of the practical uses of geometry are based upon the angle and the triangle. In order to work with the angle and the triangle we will need to understand the meanings of segment, ray, and union of sets.

A segment is a part of a line together with its endpoints.

A ray is a half-line together with its endpoint.

The union of two sets is a set consisting of all the elements of the first set together with all the elements of the second set.

Using these ideas we can now explain the meanings of angle and triangle.

An angle is a set of points consisting of two rays with an endpoint in common and not both on the same straight line.

The triangle ABC, written ΔABC, is the union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ provided that A, B, and C are not on the same straight line.

The following ideas are important in mathematics:

Separation

In this chapter, the following three basic types of separation are considered.

1. Any plane separates space into two half-spaces.
2. Any line of a plane separates the plane into two half-planes.
3. Any point of a line separates the line into two half-lines.

One-To-One Correspondence

Two sets can be shown to be in one-to-one correspondence if

a. Each element of the first set can be matched with an element of the second set, and

b. Each element of the second set can be matched with an element of the first set.

Simple Closed Curve

A curve is said to be a *simple* closed curve if it can be drawn

a. So that it starts and stops at the same point.

b. No other point is touched twice by the pencil mark.

GLOSSARY

ANGLE: An angle is a set of points consisting of two rays with an endpoint in common and not both on the same straight line.

HALF-LINE: A half-line is a ray without its endpoint.

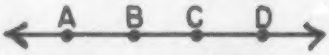
REGION: A region is a set of points in the interior of a simple closed curve.

UNION OF SETS: The union of two sets is a set consisting of all elements of the first set together with all elements of the second set.

7-7. Chapter Review.

Exercises 7-7.

1. Draw a horizonatal line. Label four points on it A, B, C, D (as shown). Name two rays



   a. Whose endpoints are B.

   b. Whose union is the line.

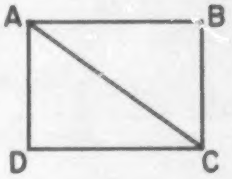
   c. Whose union is not the line but contains A, B, C, and D.

   d. Whose union does not contain A.

   e. Whose intersection is a point.

   f. Whose intersection is empty.

2. In the drawing at the right

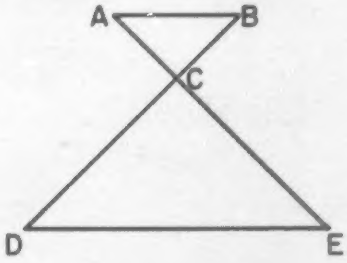


   a. Explain the difference between $\overleftrightarrow{AD}$ and $\overline{AD}$.

   b. Explain the difference between $\overline{AB}$ and $\overrightarrow{AB}$.

   c. What is the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{CD}$?

3. In the figure at the right



   a. What is $\overline{AB} \cap \overline{AC}$?

   b. What is $\overline{BC} \cup \overline{CD}$?

   c. What is $\overleftrightarrow{AE} \cap \overleftrightarrow{BD}$?

   d. What is $\overline{AB} \cap \overline{DE}$?

4. Draw a horizontal line. Label four points on it P, Q, R, and S in that order from left to right. Name two segments

   a. Whose intersection is a segment.

   b. Whose intersection is a point.

   c. Whose intersection is empty.

   d. Whose union is not a segment.

5. If $R = \{1, 3, 5, 7, 9, 10\}$
   And $S = \{4, 7, 10\}$
   Write

   a. $R \cap S$

   b. $R \cup S$

6. In the figure at the right

   a. Name a point on the B-side of $\overleftrightarrow{AC}$.

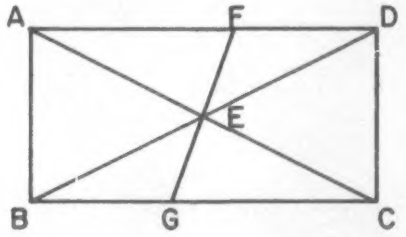
   b. Name a point on the F-side of $\overleftrightarrow{BD}$.

   c. What is $\overleftrightarrow{AC} \cap \overline{BD}$?

   d. What is $\overline{AB} \cap \overline{DC}$?

   e. What is $\overline{EF} \cup \overline{EG}$?

   f. What is $\overrightarrow{EB} \cup \overrightarrow{EG}$?



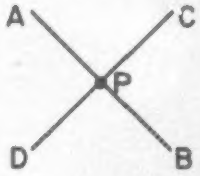
7. Explain how each of the following may be used as an example of a separation.

   a. The plane of a shelf in a bookcase.

   b. The left sideline on a basketball court.

   c. The plane of a store window.

   d. Point P.



8. Draw ΔABC.

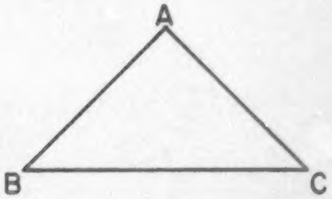
   a. Name the angle in the triangle whose vertex is B.

   b. What is $\overline{AB} \cap \overline{AC}$?

   c. Are the sides of the triangle rays, half-lines, or segments?

   d. In the drawing, extend $\overline{AB}$ in both directions to obtain $\overleftrightarrow{AB}$. What is $\overline{AB} \cap \overleftrightarrow{AB}$?

   e. What is $\overline{AB} \cap \Delta ABC$?



9. In the figure at the right, what is

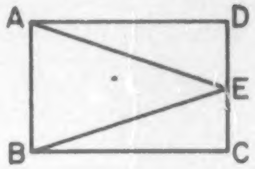
   a. $\overleftrightarrow{AE} \cap \overleftrightarrow{EB}$?

   b. $\overline{DE} \cup \overline{EC}$?

   c. $DEBC \cap \overleftrightarrow{DE}$?

   d. $\overrightarrow{EA} \cup \overrightarrow{ED}$?

   e. $\overline{AD} \cap \overline{BC}$?



10. In the diagram at the right

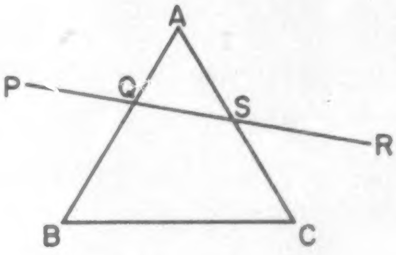
   a. Point Q separates______.

   b. C is on the________side of $\overleftrightarrow{PR}$.

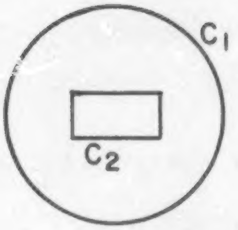
   c. $\overrightarrow{QB} \cup \overrightarrow{QS}$ is______.

   d. $\Delta ABC \cap \overleftrightarrow{PR}$ is______.

   e. $<CBA \cap \overline{PR}$ is______.



11. In the figure at the right, the two simple closed curves are represented by C_1 and C_2.



a. Make a copy of this figure.

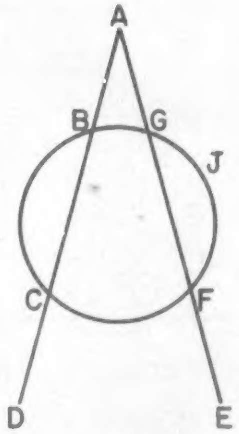
b. Shade the exterior of C_2 with vertical lines.

c. Shade the interior of C_1 with horizontal lines.

d. Using the symbol for intersection indicate the region that is doubly shaded.

12. S is the set of states in the United States.
C is the set of state capitals in the United States.
Describe a one-to-one correspondence between set C and set D.

13. In the figure at the right, J is a simple closed curve. What is



a. $\overleftrightarrow{DA} \cap \overleftrightarrow{FA}$?

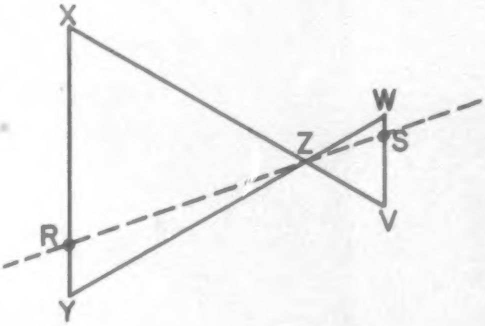
b. $J \cap \overrightarrow{AE}$?

c. A point outside J?

d. $\overrightarrow{AC} \cup \overrightarrow{AF}$?

e. $\overline{AB} \cup \overline{BC}$?

14. By drawing lines through point Z (like $\overleftrightarrow{RS}$) show how to set up a one-to-one correspondence between the set of points on $\overline{XY}$ and the set of points on $\overline{WV}$.



7-8. Cumulative Review.

Exercises 7-8.

How Are You Doing?

In exercises 1-8 answer True or False.

1. The numeral for "seven" would look the same written in base seven or base ten.
2. $0 + 0$ is meaningless.
3. The numeral following 33_{four} is 100_{four}.
4. Some odd numbers are divisible by two.
5. The difference between any two prime numbers greater than 5 is always an even number.
6. A set is closed under addition if the sum of any two elements is an element of the set.
7. A common denominator of two fractions does not have to be a multiple of the denominators of both of the given fractions.
8. If a line contains two different points of a plane, it lies in the plane.
9. 3140_{eight} is how many times as large as 314_{eight}.
10. Find a whole number which can be used for x to make the following true.

    a. $7 + x = 15$

    b. $1 + x = x + 1$

    c. $x = 0 + 3$

11. In the figure at the right

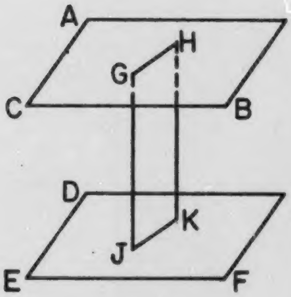
a. What is $\overleftrightarrow{GJ} \cap$ plane DEF?

b. What is $\overleftrightarrow{GH} \cap$ plane DEF?

c. What is plane ACB $\cap$ plane DEF?

d. What is $\overleftrightarrow{JK} \cap$ plane DEF?

e. What is $\overleftrightarrow{HK} \cap$ plane ACB?



12. Which of these: 2, 3, 4, 5, 6, 9, 10 is a factor of

a. 60 b. 165

13. Perform the indicated operations and simplify

a. $\frac{2}{7} + \frac{3}{7}$ c. $\frac{9}{10} - \frac{5}{6}$

b. $\frac{5}{7} \div \frac{2}{3}$ d. $\frac{3}{8} \cdot \frac{4}{7}$

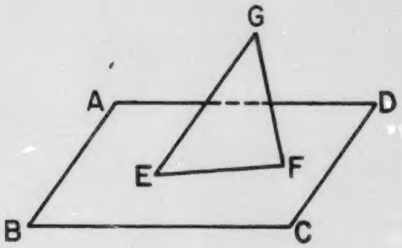
14. In the diagram on the right

a. What is plane ABC $\cap \overline{EF}$?

b. What is $\overleftrightarrow{EG} \cap$ plane ADC?

c. Explain why $\overleftrightarrow{GF}$ and $\overleftrightarrow{GE}$ are in the same plane.

d. Explain why $\overleftrightarrow{EF}$ is in the same plane as $\overleftrightarrow{GE}$ and $\overleftrightarrow{GF}$.



Chapter 8

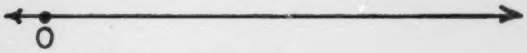
RATIONAL NUMBERS AND THE NUMBER LINE

In this chapter we will see how the rational numbers can be represented as points on a line. We will see how the operations of addition, subtraction, multiplication and division can be interpreted geometrically. This should make these operations easier to understand because we can "see" them. The idea of representing numbers as points on a line is of great importance in mathematics. This idea is used in graphing, in analytic geometry, calculus, vectors. In fact this idea is used in almost all of mathematics.

8-1. The Number Line.

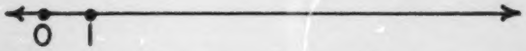
Do you remember how the number line was formed in Chapter 3? We started with a line.

Then we selected a point on the line which we called "0".

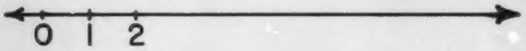


Next we chose a unit of distance

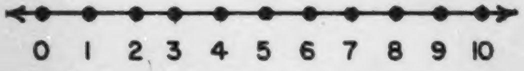
and using this unit of distance starting with 0 we marked off a point to the right of 0 which we called "1".



Starting with 1 we layed off our unit of distance to find a point which we called "2".

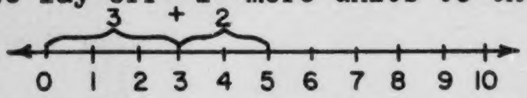


And so on:



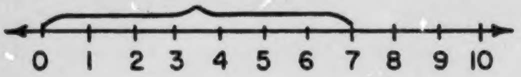
The counting number at any point shows how many segments are measured off from 0 to the point.

To add 3 and 2 on the number line we start with 0 and lay off 3 units of distance. Then we lay off 3 units of distance. Then we lay off 2 more units to the right.

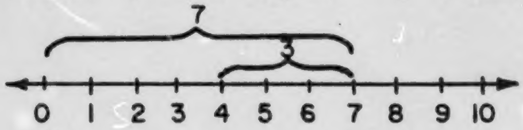


From the diagram we see that the sum is 5.

How would we subtract on the number line? Let us try to find 7 - 3. First we would lay off 7 on the number line.

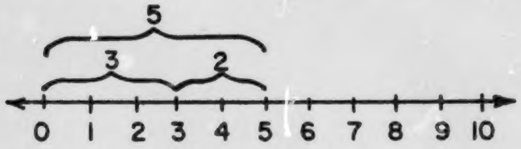


Then what? Well to add 3 to 7 we would start from 7 on the number line and lay off 3 units to the right. To take 3 away from 7 we start from 7 and lay off 3 units to the left.



We see from the diagram that the difference is 4. It is worth noting that the same diagram which shows us that

$$3 + 2 = 5$$

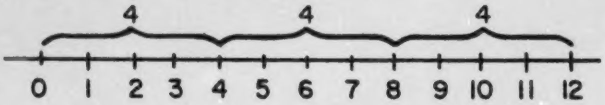


also shows us that

$$5 - 2 = 3$$

This happens because "3 + 2 = 5" and "5 - 2 = 3" are just two different ways of saying the same thing.

We can also use the number line to multiply and divide. Let us try to multiply 4 by 3. To do this we will start with 0 and lay off three segments each of length 4.



We see that

$$4 + 4 + 4 = 12$$

or

$$3 \cdot 4 = 12.$$

We can multiply other whole numbers in the same way.

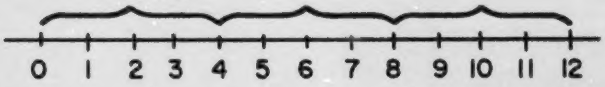
Let us now try a division problem, say

$$12 \div 3.$$

Here we wish to divide the segment from 0 to 12 into three equal parts. Then we will ask. "What is the common length of these three equal parts?" The answer to this question will be the number

$$12 \div 3.$$

Our diagram looks like this



The quotient appears here.

We observe that the same diagram which shows that

$$3 \cdot 4 = 12$$

also shows that

$$12 \div 3 = 4.$$

This happens because $3 \cdot 4 = 12$ and $12 \div 3 = 4$ are just two different ways of saying the same thing. Therefore we draw the same diagram to illustrate the two statements and interpret the diagram in two different ways.

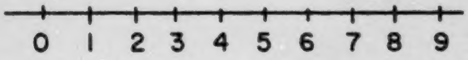
Exercises 8-1

In each of the following problems draw a copy of the number line and show how to indicate the given operation.

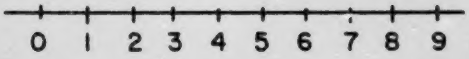
1. a. Show how to find $4 + 3$
   b. Show how to find $7 - 3$
2. a. Show how to find $3 + 4$
   b. Show how to find $7 - 4$
3. a. Show how to find $5 \cdot 2$
   b. Show how to find $10 \div 5$
4. a. Show how to find $2 \cdot 5$
   b. Show how to find $10 \div 2$

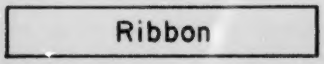
8-2. Locating Rational Numbers On The Number Line.

In the preceding section we saw how we could represent on the number line the problem of dividing 12 by 3. We did it by dividing a segment 12 units long into 3 segments of equal length. Can we divide 7 by 3? This presents some difficulties. Can you see why? The reason is that $7 \div 3$ is not a whole number and the only numbers marked on our number line are the whole numbers. But let us proceed in the same way as we did in dividing 12 by 3. Here is the number line.

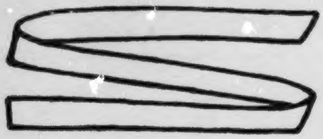


Now take a piece of ribbon (or a strip of paper) 7 units in length.

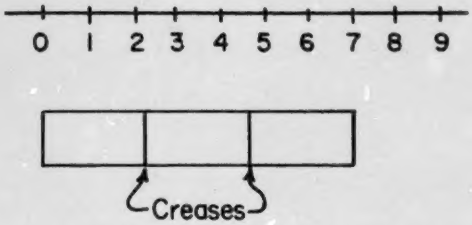




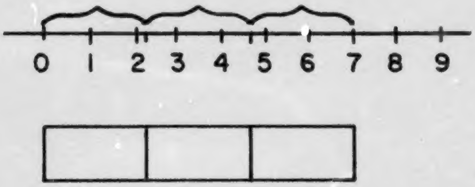
Fold the ribbon like this



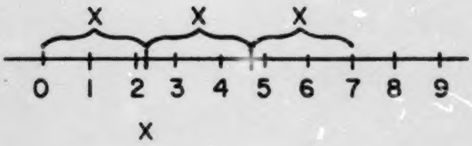
into three pieces of equal length and crease it at the folds. Now unfold the ribbon and lay it between 0 and 7 on the number line.



Now transfer the creases on to the number line.



The segment from 0 to 7 has been divided into three segments of equal length. Let us label as "x" the point where the first crease fell.



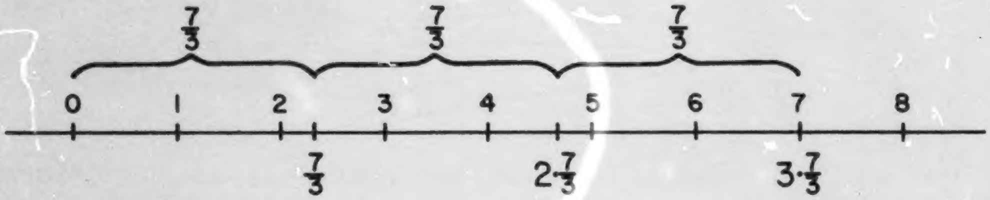
The letter x then denotes the length of the segment from 0 to the point labeled "x". Each of the three segments has length x and therefore

$$3 \cdot x = 7.$$

Other ways of saying this are

$$x = 7 \div 3 \quad \text{or} \quad x = \frac{7}{3}.$$

So we have



The diagram shows us that

$$3 \cdot \frac{7}{3} = 7.$$

In the same way we can represent every number

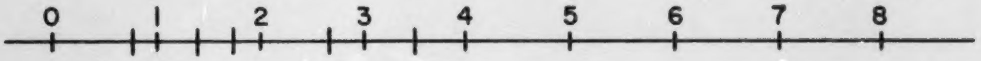
$$\frac{a}{b}$$

as a point on the line in such a way that

$$b \cdot \frac{a}{b} = a.$$

Exercises 8-2

1. Label the indicated points on the number line with the given numerals.



$\frac{7}{2}$, $\frac{5}{7}$, $\frac{7}{4}$, $\frac{8}{3}$, $\frac{7}{5}$

2. Draw a number line as in Exercise 1. Locate these numbers on the number line by folding a ribbon or strip of paper.

$\frac{3}{2}$, $\frac{2}{3}$, $\frac{6}{5}$, $\frac{7}{2}$, $\frac{5}{4}$

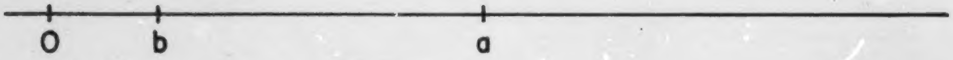
3. a. Place $\frac{5}{2}$ on the number line estimating by eye. Check to see whether the point you guessed divides the segment from 0 to 5 into 2 segments of equal length. If it does not, then alter your guess.

   b. Repeat the procedure in Part a for the numbers:

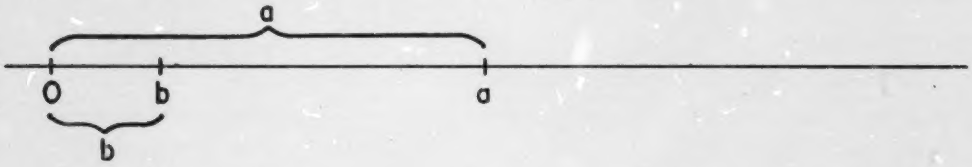
   $\frac{1}{2}$, $\frac{11}{2}$, $\frac{5}{3}$, $\frac{7}{3}$, $\frac{8}{5}$, $\frac{6}{5}$

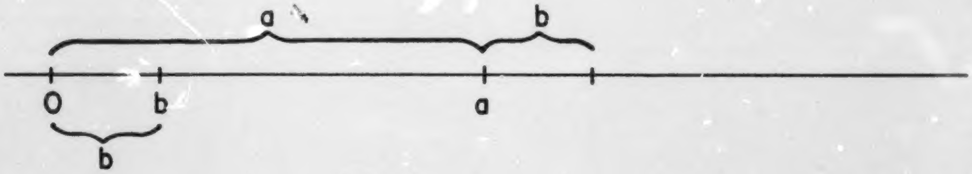
8-3. Operations on the Number Line.

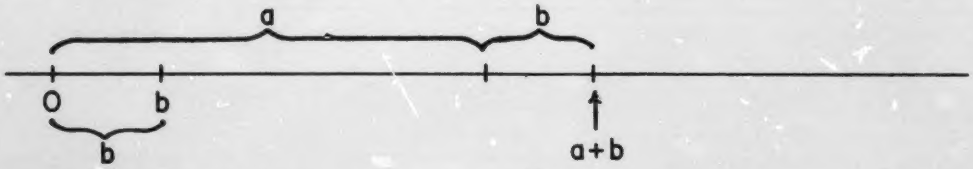
Once the rational numbers have been located on the number line they can be added and subtracted, multiplied and divided in the same way as whole numbers. Let us look at some examples to see what this means.



Here we see the number line with two rational numbers a and b located. How do we find the number a + b on the number line? Here is the method with all the steps shown.



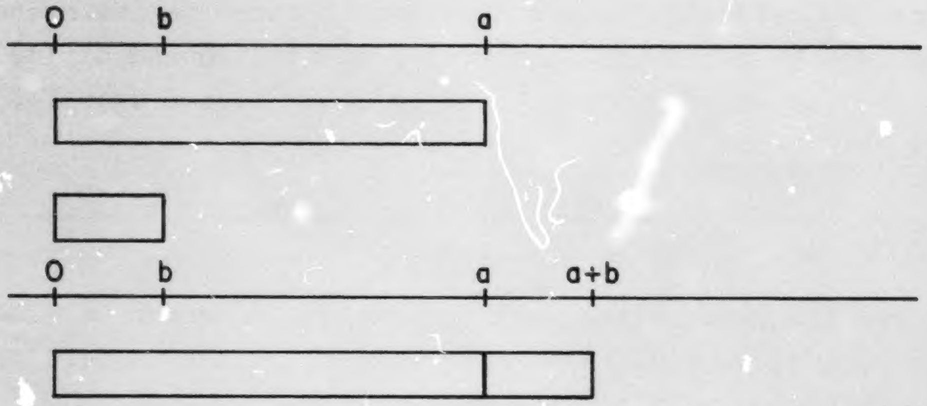




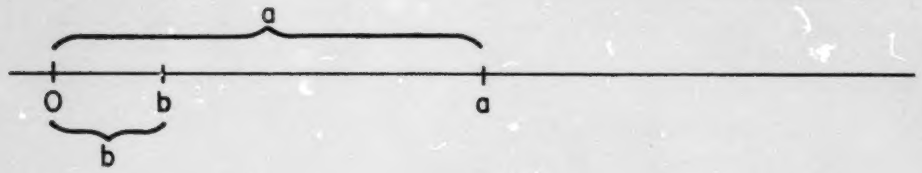
We can see from the following example that this geometrical method of adding numbers on the number line is the "right" method.

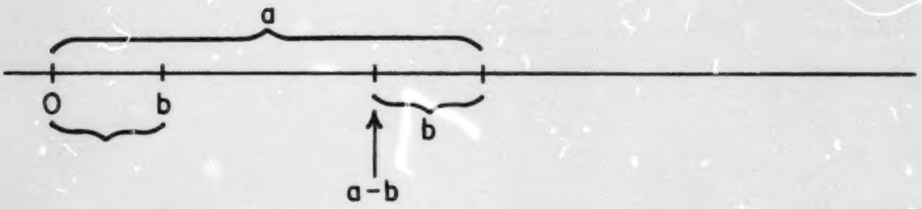
Example 1: Jean's mother gave her a ribbon a yards long and her father gave her a ribbon b yards long. How much ribbon did she have in all?

Here is the geometrical solution:



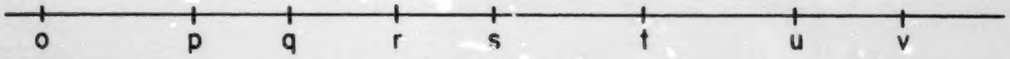
Subtraction is done in a similar way.





Class Discussion Exercises 8-3a

1. Here is the number line with certain rational numbers indicated. Answer the questions below.



Which of the indicated numbers is:

a. p + t

b. q + t

c. t - p

d. t - q

e. r + q

f. s + p

g. t - s

h. u - p

i. t - r

j. v - t

k. u - t

l. v - q

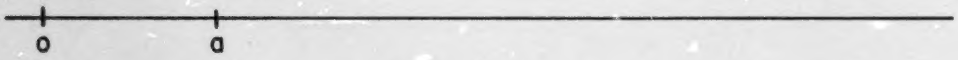
m. t + o

n. o + s

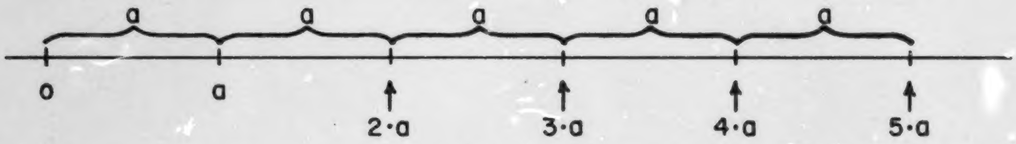
o. v - o

2. Make up a problem using ribbon like Example 1 and use it to show that the geometrical method of subtracting numbers is the "right" method.

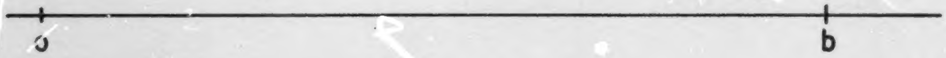
Now let us look at some examples illustrating multiplication and division by whole numbers. Here is the number line with a rational number a indicated.



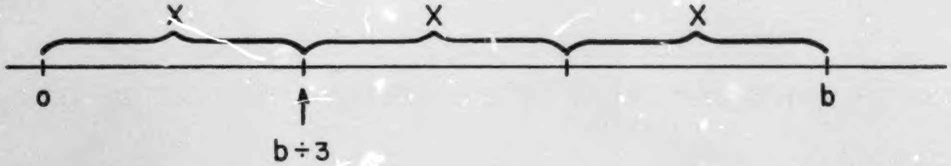
Where is $5 \cdot a$?



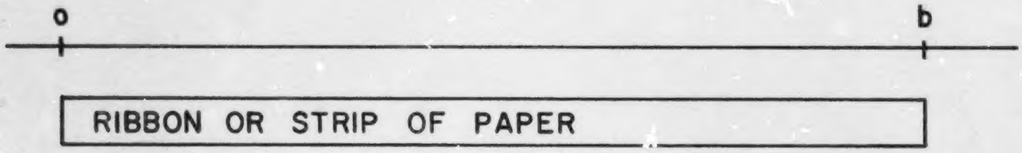
Here is the number line with the rational number b indicated.

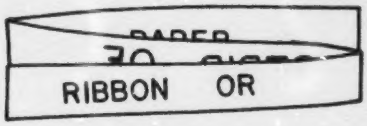


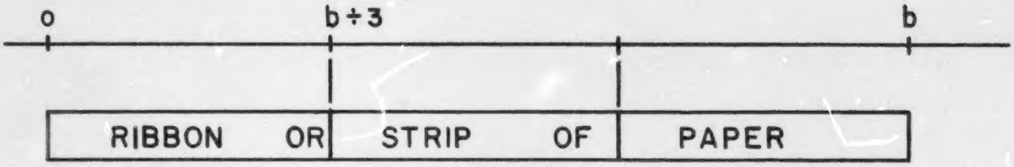
Where is $b \div 3$?



Here we split the segment from 0 to b into three segments of equal length. We could do this approximately by eye. When you take a geometry course you will see a precise geometrical method for doing this. Using what we know now we could find the $b + 3$ on the number line by folding a ribbon or strip of paper.

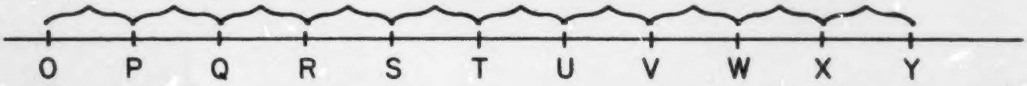

o
b
RIBBON OR STRIP OF PAPER


PAPER
RIBBON OR


o
b÷3
b
RIBBON OR
STRIP OF
PAPER

Classroom Discussion Exercises 8-3b

1. The segments indicated by the braces are equal in length.


O
P
Q
R
S
T
U
V
W
X
Y

Which of the indicated numbers is

a. $3 \cdot q$

b. $r + 3$

c. $y + 5$

d. $y + 2$

e. $2 \cdot t$

f. $w + 4$

g. $x + 3$

h. $2 \cdot r$

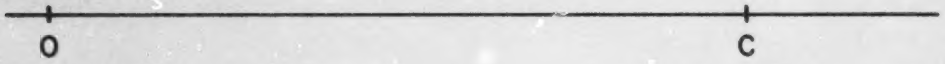
i. $7 \cdot p$

j. $w + 2$

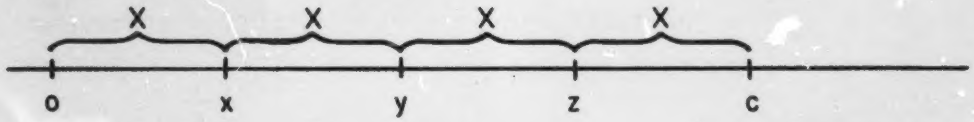
k. $1 \cdot u$

l. $u + 3$

Consider the example below; c is a rational number.



The points x, y, z split the segment from o to c into four segments of equal length.



Do you see that $x = c \div 4$? Do you see that $y = c \div 2$? Can we express z in terms of c? We see that

$$z = 3 \cdot x.$$

Since $x = c \div 4$ we see that

$$z = 3 \cdot (c \div 4).$$

In Chapter 6 we saw that

$$c \div 4 = c \cdot \frac{1}{4}$$

so that

$$z = 3 \cdot (c \cdot \frac{1}{4}).$$

Using the commutative and associative properties we can rewrite the right side of this equality to get

$$z = (3 \cdot \frac{1}{4}) \cdot c$$

or

$$z = \frac{3}{4} \cdot c$$

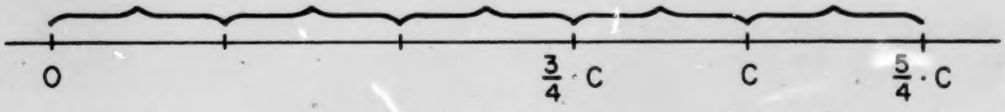
This example shows us how to multiply a rational number on the number line by any given rational number. We can tell what we did in words. To find

$$\frac{3}{4} \cdot c$$

on the number line:

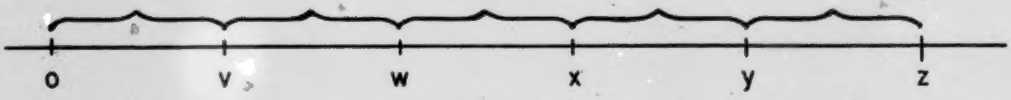
1. Split the segment from o to c into four segments of equal length.
2. Count three of these segments from the left.
3. The right hand endpoint of the third segment is $\frac{3}{4} \cdot c$.

How would you find $\frac{5}{4} \cdot c$? Do you agree that it is the point indicated below?



Classroom Discussion Exercises 8-3c

1. The segments indicated by the braces are equal in length



which of the indicated numbers is equal to

a. $\frac{3}{4} \cdot y$

b. $\frac{5}{4} \cdot y$

c. $\frac{2}{3} \cdot x$

d. $\frac{5}{3} \cdot x$

e. $\frac{1}{3} \cdot x$

f. $\frac{1}{4} \cdot y$

g. $\frac{3}{5} \cdot z$

h. $\frac{1}{5} \cdot z$

i. $\frac{5}{2} \cdot w$

j. $\frac{1}{2} \cdot y$

k. $\frac{2}{4} \cdot y$

l. $\frac{4}{3} \cdot x$

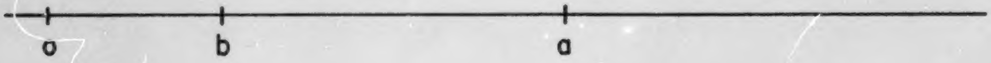
We would like to see how to do division like

$$c \div \frac{3}{5}$$

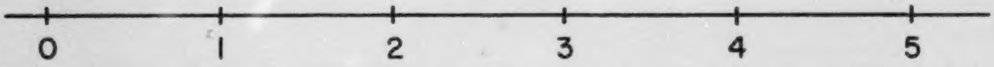
geometrically. But such problems will be easier when we know about reciprocals. We will learn about reciprocals in the next section.

Exercises 8-3d

1. With a and b as indicated below find $a + b$, $a - b$, $a \div 2$, $3 \cdot b$, $a \div 3$, $4b$

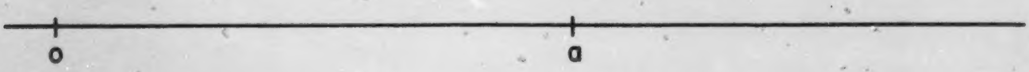


2. Locate $\frac{5}{2}$ and $\frac{4}{3}$ on the number line below.

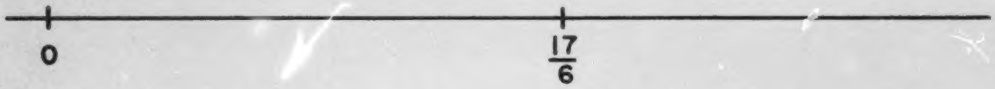


Then find $\frac{5}{2} + \frac{4}{3}$, $\frac{5}{2} - \frac{4}{3}$, $2 \cdot \frac{4}{3}$ and $\frac{5}{2} \div 2$ geometrically.

3. On the number line below place the numbers $\frac{1}{2} \cdot a$, $\frac{1}{3} \cdot a$, $\frac{2}{3} \cdot a$, $\frac{4}{3} \cdot a$, $\frac{3}{4} \cdot a$, $\frac{7}{4} \cdot a$ estimating the positions of these numbers by eye. Then check to determine how nearly right you were.



4. On the number line below, the number $\frac{17}{6}$ is indicated. Find $\frac{1}{2} \cdot \frac{17}{6}$, $\frac{1}{3} \cdot \frac{17}{6}$, $\frac{2}{3} \cdot \frac{17}{6}$, $\frac{4}{3} \cdot \frac{17}{6}$, $\frac{3}{4} \cdot \frac{17}{6}$, $\frac{7}{4} \cdot \frac{17}{6}$ estimating the position of these numbers by eye. Then check to see how nearly right you are.



8-4. Reciprocals.

Two numbers are said to be reciprocals of each other if the product of the two numbers is 1. What is the reciprocal of 6? Another way to ask this question is: What is the number x for which

$$6 \cdot x = 1?$$

We asked many questions like this one in Chapter 6. The answer is that $\frac{1}{6}$ is the number x for which $6 \cdot x = 1$. Or more simply

$$6 \cdot \frac{1}{6} = 1.$$

This last equality shows us that $\frac{1}{6}$ is the reciprocal of 6. It also shows us that 6 is the reciprocal of $\frac{1}{6}$. The numbers 6 and $\frac{1}{6}$ are reciprocals of each other.

Classroom Discussion Exercises 8-4a

1. What are the reciprocals of the given numbers.

a. 6
b. $\frac{1}{6}$
c. 8
d. $\frac{1}{8}$
e. $\frac{1}{15}$
f. 7
g. 19
h. 25
i. $\frac{1}{12}$
j. $\frac{1}{3}$
k. 1
ℓ. 255
m. $\frac{1}{1729}$
n. 65
o. 92
p. $\frac{1}{92}$

All rational numbers (except 0) have reciprocals. Can you see how to find the reciprocal of $\frac{5}{8}$? The reciprocal of $\frac{5}{8}$ is $\frac{8}{5}$ because

$$\frac{5}{8} \cdot \frac{8}{5} = \frac{5 \cdot 8}{8 \cdot 5}.$$

The fraction $\frac{5 \cdot 8}{8 \cdot 5}$ is a name for 1 because its numerator and denominator are the same. Whenever we divide a number by itself the quotient is 1.

Try another. What is the reciprocal of $\frac{7}{3}$? Was your guess $\frac{3}{7}$? If you did you are right. Check it.

$$\frac{7}{3} \cdot \frac{3}{7} = \frac{7 \cdot 3}{3 \cdot 7} = 1.$$

Perhaps you can guess a general rule. What is the reciprocal of $\frac{a}{b}$? Did you guess $\frac{b}{a}$? Let us check.

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{a \cdot b}{b \cdot a} = 1.$$

Why is $\frac{a \cdot b}{b \cdot a} = 1$? Because the numerator and denominator of the fraction

$$\frac{a \cdot b}{b \cdot a}$$

are the same.

Now let us state this as one of our rules.

Reciprocal Rule: If a and b are whole numbers different from 0 then the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

This rule tells us that the reciprocal of $\frac{7}{3}$ is $\frac{3}{7}$ and that the reciprocal of $\frac{5}{8}$ is $\frac{8}{5}$. This rule is especially simple. To find the reciprocal of a rational number expressed as a fraction we just turn the fraction over. A fancier word for "turn over" is "invert". To find the reciprocal of a rational number expressed as a fraction, just invert the fraction.

Class Discussion Exercises 8-4b

1. Find the reciprocals of the following numbers.

a. $\frac{5}{8}$
b. $\frac{7}{3}$
c. $\frac{8}{5}$
d. $\frac{3}{7}$
e. $\frac{17}{6}$
f. $\frac{9}{4}$
g. $\frac{6}{1}$
h. $\frac{1}{6}$
i. $\frac{11}{7}$
j. $\frac{2}{3}$
k. $\frac{3}{4}$
l. $\frac{6}{9}$
m. $\frac{1}{1}$
n. $\frac{14}{15}$
o. $\frac{15}{14}$
p. $\frac{1729}{1492}$

One way in which reciprocals are useful is in giving a simpler way of remembering the rule for division. Remember that

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$$

and also

$$\frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

Therefore $\frac{a}{b} \div \frac{c}{d}$ and $\frac{a}{b} \cdot \frac{d}{c}$ are both names for $\frac{a \cdot d}{b \cdot c}$ so that

$$\underset{\text{dividend}}{\frac{a}{b}} \div \underset{\text{divisor}}{\frac{c}{d}} = \underset{\text{dividend}}{\frac{a}{b}} \cdot \underset{\text{reciprocal of divisor}}{\frac{d}{c}}$$

The division rule can now be reworded as follows:

Division Rule: To divide two rational numbers, multiply the dividend by the reciprocal of the divisor.

Classroom Discussion Exercises 8-4c

1. Express the following quotients as the product of the dividend by the reciprocal of the divisor and then perform the multiplication..

Example: $\frac{2}{3} \div \frac{5}{7}$

Solution: $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$

a. $\frac{3}{2} \div \frac{7}{5}$

b. $\frac{3}{4} \div \frac{2}{3}$

c. $\frac{3}{7} \div \frac{5}{4}$

d. $\frac{3}{11} \div \frac{2}{7}$

e. $\frac{2}{9} \div \frac{3}{4}$

f. $\frac{6}{7} \div \frac{5}{3}$

g. $\frac{1}{2} \div \frac{1}{4}$

h. $\frac{1}{2} \div 4$

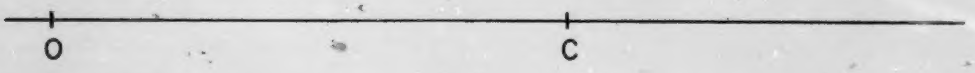
i. $\frac{3}{4} \div \frac{1}{5}$

j. $\frac{2}{3} \div \frac{3}{2}$

k. $\frac{6}{5} \div \frac{5}{6}$

l. $\frac{11}{7} \div \frac{5}{9}$

Now we can see how to divide geometrically. Here is a number c on the number line.



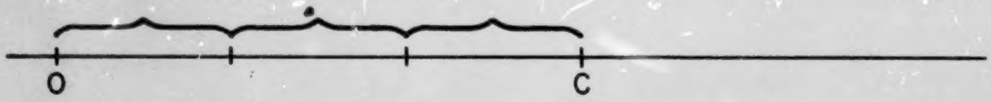
How do we find

$$c \div \frac{3}{4}?$$

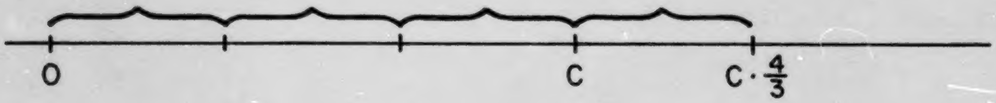
We use the fact that

$$c \div \frac{3}{4} = c \cdot \frac{4}{3}.$$

Now our problem is to multiply c by $\frac{4}{3}$. We know how to do this. We split the segment from o to c into 3 segments of equal length.



and count off four segments of this length.



Exercises 8-4d

1. Find the reciprocals of the following numbers.

   a. $\frac{6}{7}$

   b. $\frac{17}{6}$

   c. 5

   d. $\frac{1}{3}$

   e. $2 \cdot \frac{3}{5}$

   f. $3 \cdot \frac{5}{6}$

   g. $\frac{1}{2} + \frac{1}{3}$

   h. $\frac{1}{5} \cdot \frac{4}{7}$

   i. $2 + \frac{2}{3}$

   j. $\frac{5}{6} - \frac{1}{3}$

   k. $\frac{2}{3} \cdot \left(\frac{1}{2} + \frac{1}{5}\right)$

   l. $\left(\frac{1}{4} + \frac{1}{5}\right) - \frac{1}{3}$

2. Work out the indicated divisions using reciprocals.

   a. $\frac{2}{3} + \frac{4}{9}$

   b. $\frac{2}{7} + \frac{1}{3}$

   c. $1 + \frac{4}{5}$

   d. $1 + \frac{2}{3}$

   e. $\frac{5}{6} + \frac{2}{3}$

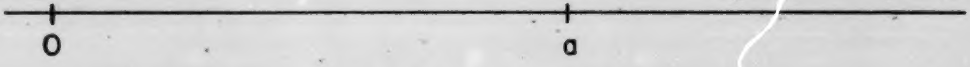
   f. $\frac{10}{3} + \frac{5}{3}$

   g. $\frac{2}{5} + \frac{3}{4}$

   h. $\left(\frac{2}{3} + \frac{1}{2}\right) + \frac{7}{5}$

   i. $\left(\frac{3}{5} + \frac{1}{3}\right) + \left(\frac{1}{2} + \frac{1}{6}\right)$

3. In the diagram below locate $a + \frac{5}{3}$, $a + \frac{3}{5}$, $a + \frac{1}{2}$, $a + \frac{4}{3}$ approximately



8-5. Comparing Rational Numbers.

Every rational number is the product of a whole number by the reciprocal of a non-zero whole number. For example

$$\frac{3}{5} = 3 \cdot \frac{1}{5}.$$

This can be done in many ways, such as

$$\frac{3}{5} = \frac{6}{10} = 6 \cdot \frac{1}{10}; \quad \frac{3}{5} = \frac{9}{15} = 9 \cdot \frac{1}{15}.$$

The numbers

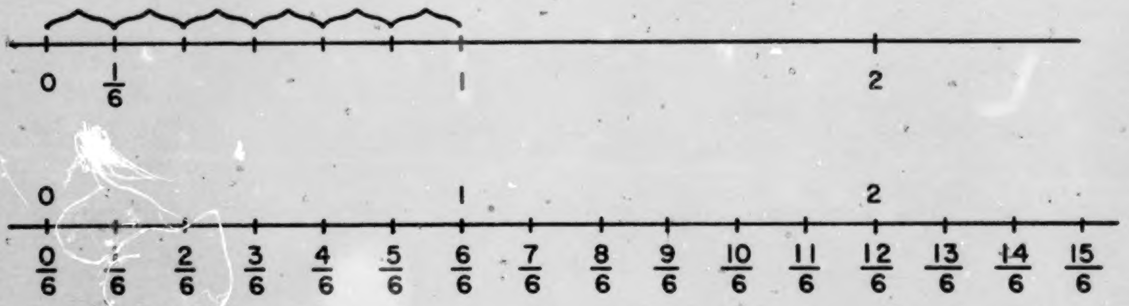
$$\frac{0}{6}, \quad \frac{1}{6}, \quad \frac{2}{6}, \quad \frac{3}{6}, \quad \frac{4}{6}, \quad \ldots.$$

are called the multiples of $\frac{1}{6}$ because

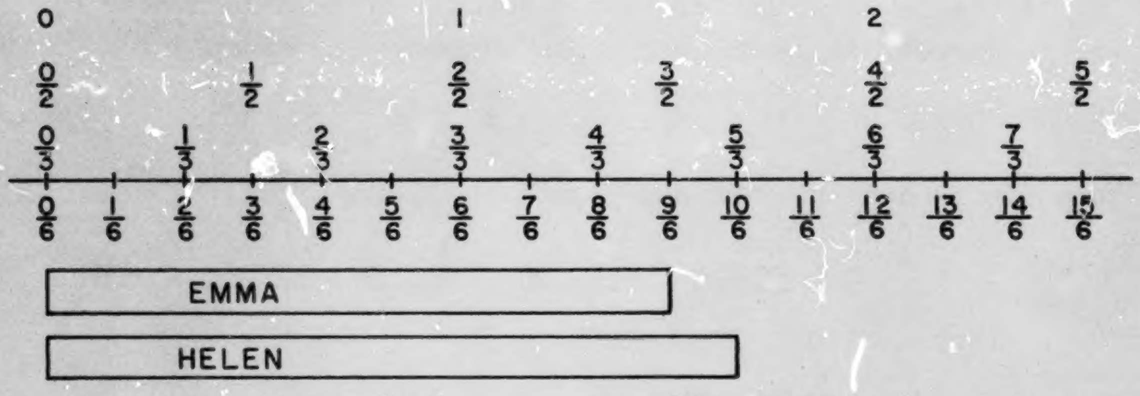
$$\frac{0}{6} = 0 \cdot \frac{1}{6}, \quad \frac{1}{6} = 1 \cdot \frac{1}{6}, \quad \frac{2}{6} = 2 \cdot \frac{1}{6}, \quad \frac{3}{6} = 3 \cdot \frac{1}{6}, \quad \ldots$$

By the "multiples" of $\frac{1}{6}$ we mean the products of $\frac{1}{6}$ by whole numbers.

Let us see where the multiples of $\frac{1}{6}$ occur on the number line.



We see that the multiples of $\frac{1}{6}$ form a set of equally spaced points numbered from left to right in order of increasing numerators. These numbers have other names some of which are indicated on the next page.


0
1
2
$\frac{0}{2}$
$\frac{1}{2}$
$\frac{2}{2}$
$\frac{3}{2}$
$\frac{4}{2}$
$\frac{5}{2}$
$\frac{0}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{3}$
$\frac{4}{3}$
$\frac{5}{3}$
$\frac{6}{3}$
$\frac{7}{3}$
$\frac{0}{6}$
$\frac{1}{6}$
$\frac{2}{6}$
$\frac{3}{6}$
$\frac{4}{6}$
$\frac{5}{6}$
$\frac{6}{6}$
$\frac{7}{6}$
$\frac{8}{6}$
$\frac{9}{6}$
$\frac{10}{6}$
$\frac{11}{6}$
$\frac{12}{6}$
$\frac{13}{6}$
$\frac{14}{6}$
$\frac{15}{6}$
EMMA
HELEN

Would you agree that $\frac{9}{6}$ is less than $\frac{10}{6}$? Suppose that Emma has a ribbon $\frac{9}{6}$ yards long and Helen has a ribbon $\frac{10}{6}$ yards long. Wouldn't you say that Emma's ribbon is shorter than Helen's ribbon? Wouldn't you say that the number $\frac{9}{6}$ is less than the number $\frac{10}{6}$? We can express this fact in symbols as

$$\frac{9}{6} < \frac{10}{6} .$$

Another way of putting this is that $\frac{10}{6}$ is greater than $\frac{9}{6}$ which we can write

$$\frac{10}{6} > \frac{9}{6} .$$

We also notice that $\frac{9}{6}$ occurs to the left of $\frac{10}{6}$ on the number line.

Two very important facts concerning comparisons of numbers are:

1. If two different fractions have the same denominator, then the fraction with the larger numerator represents the greater number.
2. Of two different numbers on the number line, the one further to the right is the greater number.

One of the most important properties of the number line is that it preserves the order of the rational numbers. That is, the rational numbers are arranged on the number line from left to right in order of increasing size.

Classroom Discussion Exercises 8-5a

1. Supply the correct symbol, $<$ or $>$ in each statement below.

Example: $\frac{7}{10} \ ? \ \frac{9}{10}$

Solution: $\frac{7}{10} < \frac{9}{10}$

a. $\frac{9}{6} \ ? \ \frac{10}{6}$

b. $\frac{11}{3} \ ? \ \frac{13}{3}$

c. $\frac{6}{5} \ ? \ \frac{4}{5}$

d. $\frac{8}{7} \ ? \ \frac{11}{7}$

e. $\frac{3}{12} \ ? \ \frac{4}{12}$

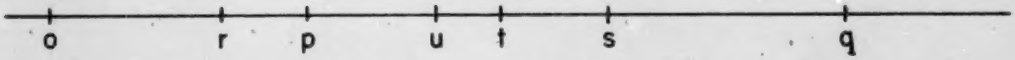
f. $\frac{6}{9} \ ? \ \frac{4}{9}$

g. $\frac{4}{19} \ ? \ \frac{22}{19}$

h. $\frac{13}{24} \ ? \ \frac{14}{24}$

i. $\frac{11}{2} \ ? \ \frac{12}{2}$

2. Certain rational numbers are indicated on the number line.



Supply the correct symbol, $<$ or $>$ in the statements below.

a. $q \ ? \ r$

b. $p \ ? \ s$

c. $t \ ? \ q$

d. $s \ ? \ u$

e. $s \ ? \ q$

f. $p \ ? \ t$

g. $u \ ? \ r$

h. $r \ ? \ t$

i. $u \ ? \ p$

How do we compare rational numbers expressed as fractions with different denominators? Which is the greater of the two numbers $\frac{3}{2}$ and $\frac{5}{3}$? If you will look back at the last figure, the one in which Emma's and Helen's ribbons were compared with the number line, you will see that "$\frac{3}{2}$" is another name for $\frac{9}{6}$ and "$\frac{5}{3}$" is another name for $\frac{10}{6}$.

Since

$$\frac{3}{2} = \frac{9}{6} \text{ and } \frac{5}{3} = \frac{10}{6}$$

and since we already know that

$$\frac{9}{6} < \frac{10}{6}$$

we see that

$$\frac{3}{2} < \frac{5}{3}.$$

The number $\frac{3}{2}$ should then occur to the left of $\frac{5}{3}$ on the number line, and we see by the last figure that this is the case.

Consider another example. Suppose that you want to compare

$$\frac{5}{7} \text{ and } \frac{2}{3}$$

to see which is the greater. How do you proceed? Well what did you do in the other example when you compared $\frac{3}{2}$ and $\frac{5}{3}$? You wrote

$$\frac{3}{2} = \frac{9}{6} \text{ and } \frac{5}{3} = \frac{10}{6}$$

and then compared $\frac{9}{6}$ and $\frac{10}{6}$. You expressed the given rational numbers $\frac{3}{2}$ and $\frac{5}{3}$ as fractions having the same denominator. We will do the same thing with $\frac{5}{7}$ and $\frac{2}{3}$. First we look for a common denominator for the given fractions. That is, we look for a common multiple of 7 and 3. We see that 21 is a common multiple of 7 and 3. And now

$$\frac{5}{7} = \frac{5 \cdot 3}{7 \cdot 3} = \frac{15}{21} \text{ and } \frac{2}{3} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{14}{21}.$$

Since $15 > 14$ we see that $\frac{15}{21} > \frac{14}{21}$ so that

$$\frac{5}{7} > \frac{2}{3}.$$

From the examples we have worked out we can see how to find a rule for comparing two rational numbers. Let us try to compare

$$\frac{a}{b} \text{ and } \frac{c}{d}.$$

Our first step will be to find a common denominator for these two fractions. We can see that $b \cdot d$ is a multiple of both b and d. We write

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d} \text{ and } \frac{c}{d} = \frac{b \cdot c}{b \cdot d}.$$

Since the fractions "$\frac{a \cdot d}{b \cdot d}$" and "$\frac{b \cdot c}{b \cdot d}$" have the same denominator, the one with the larger numerator represents the greater number. So we may write

$$\frac{a \cdot d}{b \cdot d} < \frac{b \cdot c}{b \cdot d} \quad \text{if } a \cdot d < b \cdot c$$

$$\frac{a \cdot d}{b \cdot d} > \frac{b \cdot c}{b \cdot d} \quad \text{if } a \cdot d > b \cdot c$$

$$\frac{a \cdot d}{b \cdot d} = \frac{b \cdot c}{b \cdot d} \quad \text{if } a \cdot d = b \cdot c$$

Since

$$\frac{a \cdot d}{b \cdot d} = \frac{a}{b} \text{ and } \frac{b \cdot c}{b \cdot d} = \frac{c}{d}$$

we have the rule:

Comparison Rule: If a, b, c and d are whole numbers with b and d different from 0, then

1. $\frac{a}{b} < \frac{c}{d}$ if $a \cdot d < b \cdot c$
2. $\frac{a}{b} > \frac{c}{d}$ if $a \cdot d > b \cdot c$
3. $\frac{a}{b} = \frac{c}{d}$ if $a \cdot d = b \cdot c$

Let us try this rule right away while it is still hot. Let us compare our old friends $\frac{3}{2}$ and $\frac{5}{3}$. Here we let $a = 3$, $b = 2$, $c = 5$ and $d = 3$ so that

$$\frac{a}{b} = \frac{3}{2} \text{ and } \frac{c}{d} = \frac{5}{3}.$$

Now

$$a \cdot d = 3 \cdot 3 = 9 \quad \text{and} \quad b \cdot c = 2 \cdot 5 = 10.$$

As $9 < 10$ we see that $a \cdot d < b \cdot c$ and the rule tells us that

$$\frac{a}{b} < \frac{c}{d} \quad \text{or} \quad \frac{3}{2} < \frac{5}{3}.$$

This is the answer we had found earlier in this section.

Try another example. Compare

$$\frac{9}{21} \quad \text{and} \quad \frac{6}{14}$$

Here

$$\frac{a}{b} = \frac{9}{21} \quad \text{and} \quad \frac{c}{d} = \frac{6}{14}$$

$$a \cdot d = 9 \cdot 14 = 126; \quad b \cdot c = 21 \cdot 6 = 126.$$

we have

$$a \cdot d = b \cdot c$$

and therefore

$$\frac{a}{b} = \frac{c}{d} \quad \text{or} \quad \frac{9}{21} = \frac{6}{14}.$$

Exercises 8-5b

1. Using the comparison rule, put the correct symbol, $>$, $<$ or $=$ between the given pairs of numbers.

   Example: $\frac{3}{4}$? $\frac{10}{13}$

   Solution: $3 \cdot 13 = 39$, $4 \cdot 10 = 40$, $39 < 40$ so that $\frac{3}{4} < \frac{10}{13}$.

   a. $\frac{9}{11}$? $\frac{16}{19}$
   b. $\frac{15}{4}$? $\frac{11}{3}$
   c. $\frac{10}{15}$? $\frac{8}{12}$
   d. $\frac{14}{19}$? $\frac{2}{3}$
   e. $\frac{4}{7}$? $\frac{3}{5}$
   f. $\frac{3}{17}$? $\frac{1}{5}$
   g. $\frac{11}{8}$? $\frac{4}{3}$
   h. $\frac{7}{4}$? $\frac{5}{3}$
   i. $\frac{19}{11}$? $\frac{7}{4}$
   j. $\frac{14}{6}$? $\frac{35}{15}$
   k. $\frac{9}{7}$? $\frac{5}{4}$
   ℓ. $\frac{12}{15}$? $\frac{8}{10}$

2. By expressing the two numbers as fractions having a common denominator, put the correct symbol, $>$, $<$ or $=$ between the given pairs of numbers.

Example: $\frac{2}{7} \; ? \; \frac{1}{3}$

Solution: $\frac{2}{7} = \frac{6}{21}$, $\frac{1}{3} = \frac{7}{21}$, $\frac{6}{21} < \frac{7}{21}$ so that $\frac{2}{7} < \frac{1}{3}$.

a. $\frac{3}{4} \; ? \; \frac{10}{13}$

b. $\frac{15}{4} \; ? \; \frac{18}{5}$

c. $\frac{3}{7} \; ? \; \frac{5}{12}$

d. $\frac{3}{8} \; ? \; \frac{1}{3}$

e. $\frac{4}{3} \; ? \; \frac{5}{4}$

f. $\frac{4}{6} \; ? \; \frac{6}{9}$

g. $\frac{14}{5} \; ? \; \frac{31}{11}$

h. $\frac{6}{8} \; ? \; \frac{15}{20}$

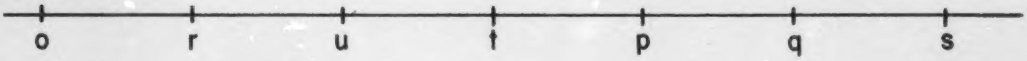
i. $\frac{2}{9} \; ? \; \frac{1}{5}$

j. $\frac{11}{7} \; ? \; \frac{8}{5}$

k. $\frac{6}{10} \; ? \; \frac{9}{15}$

l. $\frac{8}{13} \; ? \; \frac{5}{8}$

3. Certain numbers are indicated on the number line.



Supply the correct symbol, $>$ or $<$ in the statements below.

a. r ? t

b. p ? q

c. s ? p

d. p ? u

e. t ? q

f. q ? r

g. u ? q

h. p ? s

i. p ? r

4. If $a < b$ and $b < c$, then a is to the left of b on the number line and c is to the right of b so that a is to the left of c or $a < c$. If, however, $a < b$ and $c < b$, then we know that a and c are both to the left of b, but we do not know which of the two numbers a and c is farther to the left. So we cannot conclude which is the greater of a and c. In each of the following problems tell whether $a < c$, $a > c$ or no conclusion possible.

a. $a < b$ and $b < c$

b. $a > b$ and $b > c$

c. $a < b$ and $c > b$

d. $a < b$ and $c < b$

e. $a < b$ and $b > c$

f. $a > b$ and $c > b$

g. $a > b$ and $c < b$

h. $a < b$ and $c = b$

i. $a = b$ and $c < b$

j. $a > b$ and $b < c$

5. Using the method of Problem 4, replace the ? by $>$ or $<$ or write no conclusion.

a. $\frac{2}{3} < \frac{3}{4}$ and $\frac{3}{4} < \frac{5}{6}$ therefore $\frac{2}{3}$? $\frac{5}{6}$

b. $\frac{3}{7} > \frac{2}{5}$ and $\frac{3}{7} < \frac{4}{9}$ therefore $\frac{2}{5}$? $\frac{4}{9}$

c. $\frac{3}{7} > \frac{2}{5}$ and $\frac{3}{8} < \frac{3}{7}$ therefore $\frac{2}{5}$? $\frac{3}{8}$

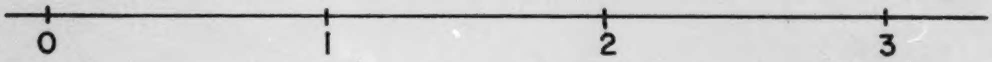
d. $\frac{2}{5} < \frac{3}{5}$ and $\frac{3}{5} < \frac{3}{4}$ therefore $\frac{3}{4}$? $\frac{2}{5}$

e. $\frac{219}{539} < \frac{1}{2}$ and $\frac{304}{607} > \frac{1}{2}$ therefore $\frac{219}{539}$? $\frac{304}{607}$

f. $\frac{607}{606} > 1$ and $\frac{418}{419} < 1$ therefore $\frac{607}{606}$? $\frac{418}{419}$

6. Work Problem 5 using the comparison rule.

7. a. On the number line below find $\frac{3}{2}$, $\frac{4}{3}$, $\frac{9}{5}$ and the reciprocals of these numbers.



b. Replace the question marks by the correct symbol $<$ or $>$.

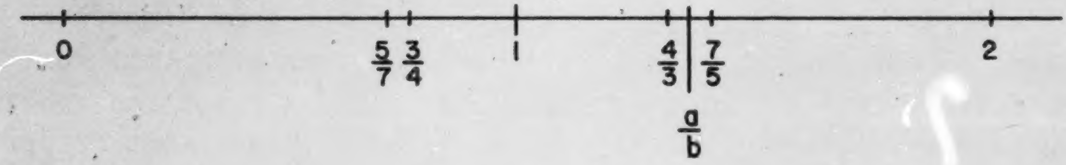
$\frac{3}{2}$? $\frac{4}{3}$; $\frac{2}{3}$? $\frac{3}{4}$

$\frac{3}{2}$? $\frac{9}{5}$; $\frac{2}{3}$? $\frac{5}{9}$

$\frac{4}{3}$? $\frac{9}{5}$; $\frac{3}{4}$? $\frac{5}{9}$

c. From Part b above can you guess the right symbol $<$ or $>$ to replace ? in the general statement, "If $\frac{a}{b} < \frac{c}{d}$ then $\frac{b}{a}$? $\frac{d}{c}$"?

8. Here is the number line with $\frac{a}{b}$ indicated. Find the approximate location of $\frac{b}{a}$.



8-6. Mixed Numbers.

We know that if the numerator and denominator of a fraction are the same, as in

$$\frac{6}{6},$$

then the fraction is a numeral for 1. If the numerator is less than the denominator, as in

$$\frac{5}{6},$$

then the fraction represents a number which is less than 1. Do you remember that we learned in the last section that

$$\frac{5}{6} < \frac{6}{6}?$$

If the numerator is greater than the denominator, as in

$$\frac{7}{6},$$

then the fraction represents a number greater than 1. For example:

$$\frac{7}{6} > \frac{6}{6}.$$

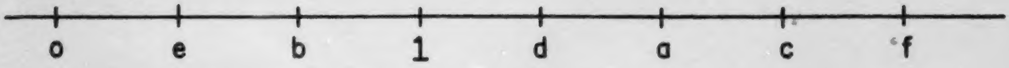
Fractions in which the numerator is less than the denominator are commonly called proper fractions. And fractions in which the numerator is either greater than or equal to the denominator are called improper fractions. Unfortunately the name "improper" seems to suggest that there is something wrong with this sort of fraction. This is not really the case. These fractions are just as good as any other fractions. You should notice that proper fractions are names for numbers which are less than 1, while improper fractions are names for numbers which are either equal to 1 or greater than 1.

Classroom Discussion Exercises 8-6a

1. Tell in each case whether the given fraction is proper or improper.

   a. $\frac{7}{5}$
   b $\frac{19}{18}$
   c. $\frac{3}{4}$
   d. $\frac{6}{2}$
   e. $\frac{8}{4}$
   f. $\frac{2}{7}$
   g. $\frac{7}{2}$
   h. $\frac{4}{9}$
   i. $\frac{9}{4}$
   j. $\frac{33}{34}$
   k. $\frac{34}{33}$
   ℓ. $\frac{9}{9}$
   m. $\frac{2}{2}$
   n. $\frac{2}{5}$
   o. $\frac{4}{10}$

2. Certain numbers are located on the number line.



Tell which of these numbers will have names that are proper fractions and which will have names that are improper fractions.

Every improper fraction can be expressed either as a whole number or as the sum of a whole number and a proper fraction. Both of these possibilities are illustrated by these examples

$$\frac{12}{6} = 2$$

$$\frac{13}{6} = \frac{12}{6} + \frac{1}{6} = 2 + \frac{1}{6}$$

It is customary to use the expression

$$2\frac{1}{6}$$

as a short way of writing

$$2 + \frac{1}{6}.$$

Similarly

$$3\frac{5}{7} = 3 + \frac{5}{7}, \quad 13\frac{2}{3} = 13 + \frac{2}{3}.$$

Such expressions as

$$2\frac{1}{6}, \quad 3\frac{5}{7}, \quad 13\frac{2}{3}$$

are commonly called "mixed numbers". This is a very bad name because it is not the numbers which are mixed, but the numerals. We know that

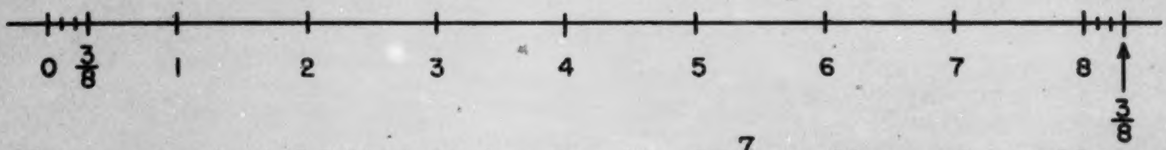
$$2\frac{1}{6} = \frac{13}{6},$$

but we do not say that $\frac{13}{6}$ is a mixed number. When we talk about mixed numbers we are talking about the things we write. But we do not write numbers, we write numerals. We see that it might be better to refer to expressions like

$$2\frac{1}{6}, \quad 3\frac{5}{7}, \quad 13\frac{2}{3}$$

as "mixed numerals" rather than "mixed numbers". But we will speak of "mixed numbers" because this expression is the one commonly used.

It is often convenient to express rational numbers as mixed numbers rather than as improper fractions. This is particularly true when making measurements. For example we could say that a soda straw is $8\frac{3}{8}$ inches long or we could say that it is $\frac{67}{8}$ inches long. The numbers $8\frac{3}{8}$ and $\frac{67}{8}$ are the same. But when we say that the straw is $8\frac{3}{8}$ inches long we have a clearer idea of how long it is. We know at once that it is more than 8 and less than 9 inches long. We can place this number on the number line more quickly when it is expressed as $8\frac{3}{8}$.



When a man is told that he has bought $13\frac{7}{10}$ gallons of gasoline he has a better idea of the quantity of gasoline than when he is told he has bought $\frac{137}{10}$ gallons of gasoline.

If a woman reading a recipe found that a cake required $\frac{19}{8}$ cups of flour what should she do? She might take 19 cups of flour and split this quantity of flour into 8 equal parts. Or she might measure out $\frac{1}{8}$ cup of flour 19 times. In the first case she was thinking

$$\frac{19}{8} = 19 \div 8.$$

In the second case she was thinking

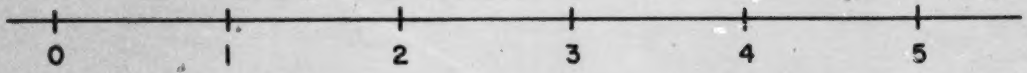
$$\frac{19}{8} = 19 \cdot \frac{1}{8}.$$

Both of these methods are correct but neither is convenient. If the recipe called for $2\frac{3}{8}$ cups of flour (the same amount of flour as above), the woman would probably do the sensible thing. She would measure 2 cups and then measure $\frac{3}{8}$ cup of flour.

In other cases, particularly in multiplication and division problems it is better to express numbers as improper fractions rather than as mixed numbers. We must be familiar with both ways of expressing numbers and we must be able to switch from one way to the other.

Exercises 8-6b

1. Copy the number line as shown below.



Locate on the number line the numbers

a. $2\frac{3}{4}$

b. $4\frac{3}{5}$

c. $1\frac{2}{3}$

d. $3\frac{1}{3}$

e. $\frac{14}{3}$

f. $\frac{11}{5}$

g. $\frac{7}{2}$

h. $\frac{15}{4}$

i. $4\frac{2}{3}$

j. $2\frac{1}{5}$

k. $3\frac{1}{2}$

ℓ. $3\frac{3}{4}$

Suppose a rational number is expressed as a mixed number and we wish to express it instead as an improper fraction. We proceed as in the following example.

Example 1: Express $6\frac{4}{5}$ as an improper fraction.

Solution: $6\frac{4}{5} = 6 + \frac{4}{5} = \frac{30}{5} + \frac{4}{5} = \frac{34}{5}$.

To express an improper fraction as a mixed numeral we may proceed as follows:

$$\frac{17}{6} = \frac{12 + 5}{6} = \frac{12}{6} + \frac{5}{6} = 2 + \frac{5}{6} = 2\frac{5}{6}.$$

The idea is to find the largest multiple of the denominator which is less than the numerator. In this case we saw that 12 was the largest multiple of 6 which is less than 17. This is a fairly easy method when the numerator and denominator of the given fraction are small. When the numerator and denominator are large, as in

$$\frac{2495}{47},$$

then we must do something else. Here is our work

$$\begin{array}{r@{}r} & 53 \\ \cline{2-2} 47 \big) & 2495 \\ & \underline{235} \\ & 145 \\ & \underline{141} \\ & 4 \end{array}$$

Now we can say that

$$\frac{2495}{47} = 53\frac{4}{47}$$

for the following reason. When we say that dividing 2495 by 47 gives a quotient of 53 and a remainder of 4, what we mean is that

$$2495 = 53 \cdot 47 + 4.$$

Since 2495 and $53 \cdot 47 + 4$ are the same number, then

$$\frac{2495}{47} = \frac{53 \cdot 47 + 4}{47}$$

since we are dividing the same number by 47 in both cases. Now

$$\frac{2495}{47} = \frac{53 \cdot 47 + 4}{47} = \frac{53 \cdot 47}{47} + \frac{4}{47} = 53 + \frac{4}{47} = 53\frac{4}{47}.$$

When working problems like this we do not show all these steps; we merely work out the division and write down the answer.

To do addition problems involving mixed numbers, add the "whole number parts" and add the "fractional parts" and then add the results.

$$6\frac{1}{2} + 4\frac{1}{3} = \left(6 + \frac{1}{2}\right) + \left(4 + \frac{1}{3}\right) = \left(6 + 4\right) + \left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= 10 + \left(\frac{3}{6} + \frac{2}{6}\right) = 10 + \frac{5}{6} = 10\frac{5}{6}$$

Sometimes the sum of the fractional parts is greater than 1. Then another step is necessary.

$$2\tfrac{3}{5} + 4\tfrac{2}{3} = (2 + 4) + \left(\frac{3}{5} + \frac{2}{3}\right) = 6 + \left(\frac{9}{15} + \frac{10}{15}\right)$$

$$= 6 + \frac{19}{15} = 6 + 1\tfrac{4}{15} = 7\tfrac{4}{15}.$$

The work is often arranged like this

$$\begin{array}{ccccc} 2\tfrac{3}{5} & = & 2\tfrac{9}{15} & & \\ \underline{4\tfrac{2}{3}} & = & \underline{4\tfrac{10}{15}} & & \\ & & 6\tfrac{19}{15} & = & 7\tfrac{4}{15} \end{array}$$

When subtracting, this same way of arranging our work is convenient.

$$\begin{array}{ccc} 4\tfrac{2}{3} & = & 4\tfrac{10}{15} \\ \underline{2\tfrac{3}{5}} & = & \underline{2\tfrac{9}{15}} \\ & & 2\tfrac{1}{15} \end{array}$$

Sometimes in subtraction problems it is necessary to "borrow". Consider

$$4\tfrac{2}{5} - 2\tfrac{5}{7}.$$

We arrange our work like this:

$$\begin{array}{ccc} 4\tfrac{2}{5} & = & 4\tfrac{14}{35} \\ \underline{2\tfrac{5}{7}} & = & \underline{2\tfrac{25}{35}} \end{array}$$

Now we see that $\frac{14}{35} < \frac{25}{35}$ so that we cannot substract as things stand now. We could express both $4\frac{2}{5}$ and $2\frac{5}{7}$ as improper fractions and subtract. Or we can proceed as follows:

$$4\frac{14}{35} \underset{\uparrow}{=} 3 + 1 + \frac{14}{35} \underset{\uparrow}{=} 3 + \frac{35}{35} + \frac{14}{35} \underset{\uparrow}{=} 3\frac{49}{35}$$

first "borrow" then express finally combine
1 from the 4 1 as $\frac{35}{35}$ the last two terms

Now we have:

$$\begin{array}{rcr} 4\frac{2}{5} & = & 3\frac{49}{35} \\ 2\frac{5}{7} & = & 2\frac{25}{35} \\ \hline & & 1\frac{24}{35} \end{array}$$

Sometimes in addition or subtraction problems you may find it more convenient to first express the numbers as improper fractions

$$4\frac{2}{3} + 2\frac{3}{5} = \frac{14}{3} + \frac{13}{5} = \frac{70}{15} + \frac{39}{15} = \frac{109}{15} = 7\frac{4}{15}.$$

$$4\frac{2}{3} - 2\frac{3}{5} = \frac{14}{3} - \frac{13}{5} = \frac{70}{15} - \frac{39}{15} = \frac{31}{15} = 2\frac{1}{15}.$$

In multiplication and division problems it is always easier to first express the numbers as improper fractions.

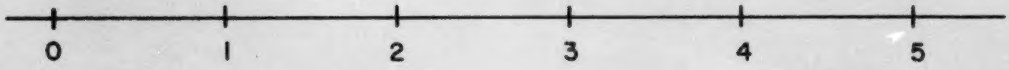
$$4\frac{2}{3} \cdot 2\frac{3}{5} = \frac{14}{3} \cdot \frac{13}{5} = \frac{14 \cdot 13}{3 \cdot 5} = \frac{182}{15} = 12\frac{2}{15}$$

$$4\frac{2}{3} \div 2\frac{3}{5} = \frac{14}{3} \div \frac{13}{5} = \frac{14}{3} \cdot \frac{5}{13} = \frac{14 \cdot 5}{3 \cdot 13} = \frac{70}{39} = 1\frac{31}{39}$$

Exercises 8-6c

1. Find the given mixed numbers on the number line below (by eye).

   a. $2\frac{1}{2}$ b. $3\frac{2}{3}$ c. $4\frac{3}{5}$ d. $2\frac{4}{5}$ e. $3\frac{2}{7}$ f. $4\frac{1}{3}$



2. Express the following mixed numbers as improper fractions.

a. $1\frac{3}{4}$

b. $2\frac{1}{2}$

c. $3\frac{2}{3}$

d. $2\frac{3}{5}$

e. $4\frac{3}{5}$

f. $3\frac{2}{7}$

g. $4\frac{1}{3}$

h. $6\frac{5}{9}$

i. $8\frac{3}{8}$

j. $10\frac{5}{7}$

k. $5\frac{5}{12}$

ℓ. $11\frac{3}{5}$

m. $14\frac{2}{7}$

n. $18\frac{1}{8}$

o. $21\frac{1}{21}$

p. $19\frac{1}{19}$

3. Express the following improper fractions as mixed numbers.

a. $\frac{7}{6}$

b. $\frac{14}{3}$

c. $\frac{13}{2}$

d. $\frac{9}{4}$

e. $\frac{27}{7}$

f. $\frac{42}{5}$

g. $\frac{65}{3}$

h. $\frac{47}{4}$

i. $\frac{167}{16}$

j. $\frac{435}{14}$

k. $\frac{229}{21}$

ℓ. $\frac{395}{151}$

m. $\frac{409}{408}$

n. $\frac{956}{222}$

o. $\frac{356}{209}$

p. $\frac{555}{77}$

4. Perform the indicated operations.

a. $1\frac{1}{4} + 2\frac{1}{2}$

b. $2\frac{2}{5} + 3\frac{1}{3}$

c. $3\frac{2}{7} + 1\frac{1}{5}$

d. $4\frac{1}{2} - 2\frac{1}{3}$

e. $3\frac{2}{5} - 1\frac{1}{4}$

f. $4\frac{1}{5} + 3\frac{1}{6}$

g. $6\frac{1}{2} - 4\frac{2}{3}$

h. $5\frac{2}{5} - 4\frac{1}{2}$

i. $6\frac{2}{3} + 4\frac{1}{2}$

j. $5\frac{4}{5} + 2\frac{3}{7}$

k. $4\frac{5}{9} - 2\frac{2}{3}$

ℓ. $11\frac{1}{3} - 6\frac{7}{8}$

5. Perform the indicated operations.

a. $2\frac{1}{2} \cdot 3\frac{1}{3}$

b. $\frac{3}{4} \cdot 4\frac{1}{3}$

c. $5\frac{2}{3} \cdot 3\frac{3}{4}$

d. $6\frac{2}{7} \div 2\frac{1}{2}$

e. $6\frac{3}{5} \div 8\frac{1}{3}$

f. $\frac{3}{4} \div 1\frac{3}{4}$

g. $4\frac{2}{3} \div 2\frac{1}{5}$

h. $3\frac{3}{5} \div 5\frac{1}{2}$

i. $\frac{14}{5} \cdot 2\frac{2}{7}$

8-7. Complex Fractions.

You will often see expressions like this

$$\frac{\frac{5}{3}}{\frac{2}{7}}.$$

(Notice that the middle line is longer than the other two lines.) This is a fraction in which the numerator and denominator themselves are fractions. What does this mean? You probably remember that when a and b are whole numbers then

$$\frac{a}{b}$$

stands for the result of dividing a by b. That is

$$\frac{a}{b} = a \div b.$$

Don't you think that $\frac{a}{b}$ should be defined in the same way when a and b are fractions? If you agree to this then you will agree that

$$\frac{\frac{5}{3}}{\frac{2}{7}}$$

should be equal to

$$\frac{5}{3} \div \frac{2}{7}.$$

Such expressions as

$$\frac{\frac{5}{3}}{\frac{2}{7}}$$

are called complex fractions.

The main thing that we want to learn about complex fractions is how to express them as ordinary fractions. We are able to do this because we know how to divide rational numbers. We show the work:

$$\frac{\frac{5}{3}}{\frac{2}{7}} = \frac{5}{3} \div \frac{2}{7} = \frac{5}{3} \cdot \frac{7}{2} = \frac{5 \cdot 7}{3 \cdot 2} = \frac{35}{6}.$$

The same method applies to fractions like

$$\frac{5}{\frac{2}{3}}.$$

we have

$$\frac{5}{\frac{2}{3}} = 5 \div \frac{2}{3} = 5 \cdot \frac{3}{2} = \frac{15}{2}.$$

Fractions of this sort in which the numerator is 1 are particularly interesting. For example

$$\frac{1}{\frac{5}{6}} = 1 \div \frac{5}{6} = 1 \cdot \frac{6}{5} = \frac{6}{5}.$$

And $\frac{6}{5}$ is the reciprocal of $\frac{5}{6}$. In the same way, if a and b are different from 0 we have

$$\frac{1}{\frac{a}{b}} = 1 \div \frac{a}{b} = 1 \cdot \frac{b}{a} = \frac{b}{a}.$$

And $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$. In Section 4 we saw that if x is a counting number then $\frac{1}{x}$ is the reciprocal of x. Now we see that if x is any rational number (except 0) then $\frac{1}{x}$ is the reciprocal of x.

Exercises 8-7

1. Simplify

a. $\dfrac{\frac{2}{3}}{\frac{3}{4}}$ d. $\dfrac{\frac{2}{5}}{\frac{3}{4}}$ g. $\dfrac{\frac{4}{9}}{\frac{2}{3}}$

b. $\dfrac{\frac{6}{7}}{\frac{3}{5}}$ e. $\dfrac{\frac{4}{3}}{\frac{3}{4}}$ h. $\dfrac{\frac{15}{8}}{\frac{5}{6}}$

c. $\dfrac{\frac{1}{3}}{\frac{1}{5}}$ f. $\dfrac{\frac{6}{5}}{\frac{3}{10}}$ i. $\dfrac{\frac{7}{8}}{\frac{9}{10}}$

2. Simplify. (Hint: first express both numerator and denominator as fractions.)

a. $\dfrac{1\frac{1}{2}}{2\frac{1}{4}}$ d. $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{4} + \frac{1}{5}}$ g. $\dfrac{2\frac{3}{4}}{\frac{2}{3} + \frac{1}{6}}$

b. $\dfrac{3\frac{1}{5}}{4\frac{2}{3}}$ e. $\dfrac{\frac{2}{3} - \frac{3}{5}}{\frac{3}{4} - \frac{2}{3}}$ h. $\dfrac{\frac{1}{3} + 2\frac{1}{2}}{\frac{1}{2} + 2\frac{1}{3}}$

c. $\dfrac{2\frac{1}{3}}{2\frac{4}{5}}$ f. $\dfrac{\frac{2}{7} + \frac{3}{8}}{\frac{4}{5} + \frac{5}{3}}$ i. $\dfrac{\frac{3}{10} + \frac{2}{3}}{\frac{4}{5} + \frac{1}{6}}$

8-8. Summary.

In this chapter we have seen how to place any rational number on the number line. We have also seen how we can add and substract, multiply and divide geometrically. Our method of placing numbers on the number line perserves the natural order of the numbers. That is if $a < b$ then a is to the left of b

on the number line. The

> Comparison Rule is a rule for telling which of two rational numbers is the larger when they are both expressed as fractions. One important part of this rule gives us a quick way of telling when two numbers are equal. That is $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$.

Two numbers whose product is 1 are said to be reciprocals of each other. Reciprocals are useful in division problems. If a first number is divided by a second number then the result is the same as if the first number is multiplied by the reciprocal of the second.

Fractions in which the numerator is less than the denominator are called proper fractions. Fractions in which the numerator is greater than the denominator are called improper fractions. Therefore proper fractions are names for numbers which are less than 1 while improper fractions are names for numbers which are greater than 1. Mixed numbers are numerals such as $2\frac{3}{7}$. This last number is short for $2 + \frac{3}{7}$. Mixed numbers are a way of expressing rational numbers as the sum of a whole number and a fraction less than 1. Mixed numbers are better than improper fractions for making measurements or for quickly estimating the size of the number. Improper fractions are easier than mixed numbers in multiplication and division problems.

Complex fractions are fractions in which the numerator or the denominator or both are themselves fractions. The complex fraction $\frac{\frac{2}{3}}{\frac{5}{7}}$ stands for the same number as $\frac{2}{3} + \frac{5}{7}$ does. To simplify a complex fraction we perform the indicated division and express the number as an ordinary fraction.

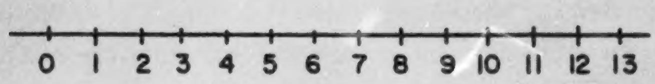
8-9. Review.

Exercises 8-9

1. On the number line below show how to find

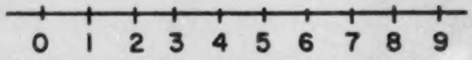
a. $4 \cdot 3$ c. $5 + 2$

b. $12 \div 4$ d. $7 - 2$

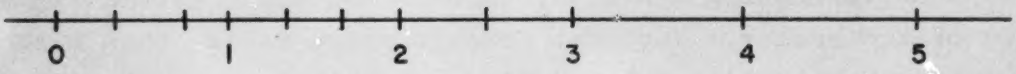


2. By folding a ribbon or strip of paper, locate the given numbers on the number line.

a. $8 \div 3$ b. $7 \div 2$

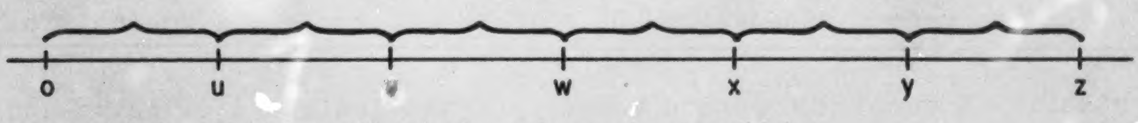


3. Label the indicated points on the number line with the given numerals.



$\frac{4}{3}$, $\frac{5}{2}$, $\frac{1}{3}$, $\frac{5}{3}$, $\frac{3}{4}$

4. The segments indicated by braces are equal in length.



which of the indicated numbers is equal to

a. $w + v$ d. $3 \cdot v$ g. $\frac{2}{3} \cdot z$ j. $w \div 3$

b. $z - x$ e. $\frac{1}{2} \cdot x$ h. $\frac{4}{3} \cdot w$ k. $v \div \frac{1}{2}$

c. $2 \cdot v$ f. $\frac{3}{4} \cdot x$ i. $\frac{5}{2} \cdot v$ ℓ. $x \div \frac{2}{3}$

5. Find the reciprocals of the given numbers.

a. 5

b. $\frac{1}{3}$

c. $\frac{2}{5}$

d. $\frac{7}{9}$

e. 1

f. $\frac{18}{7}$

g. $\frac{7}{18}$

h. $\frac{3}{22}$

i. $\frac{22}{3}$

j. $\frac{3}{2} - \frac{1}{3}$

k. $\left(\frac{2}{3} + \frac{1}{5}\right) + \frac{2}{3}$

ℓ. $\left(\frac{2}{3} + \frac{1}{6}\right) \cdot \left(\frac{1}{3} + \frac{1}{2}\right)$

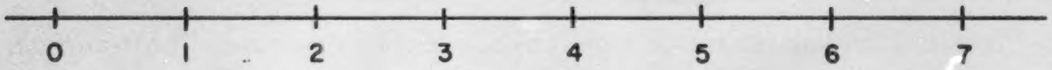
6. On the number line below, locate

a. the number 6 and its reciprocal

b. the number $\frac{3}{4}$ and its reciprocal

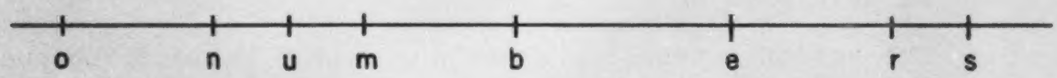
c. the number $2\frac{1}{2}$ and its reciprocal

d. the number $\frac{1}{5}$ and its reciprocal



7. In Problem 6 you can see that in each case the number 1 lies between the given number and its reciprocal. Can you tell why the number 1 will always lie between a given number and its reciprocal.

8. Certain rational numbers are indicated on the number line.



Replace the question marks by the correct symbol, $>$ or $<$.

a. b ? e

b. e ? m

c. m ? n

d. n ? r

e. r ? s

f. s ? u

14. **BRAINBUSTER.**

a. Simplify $1 + \cfrac{1}{1 + \cfrac{1}{2}}$

b. Simplify $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$

c. Simplify $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$

8-10. Cumulative Review.

Complete the statements in Exercises 1-5 below.

1. Zero is the identity for ______________.
2. Division is the __________ of multiplication.
3. The following shows a one-to-one correspondence between the _____________ numbers and the ___________ numbers.
4. The inverse operation of addition is ____________.
5. The number of counting numbers between 4 and 5 is _____.
6. In each case, what are the elements of the set that is the intersection of the given sets.
   a. The set of even numbers less than 25 and the set of of multiples of 3.
   b. The set of baseball players now in the National League and the set of baseball players now in the American League.
7. List the multiples of 4 which are greater than 30 and less than 50.
8. What is the least common multiple of 2, 5, and 7?

9. Perform the indicated operation and simplify.

a. $\left(\frac{1}{3} + \frac{1}{2}\right) + \frac{1}{6}$

b. $\left(\frac{7}{10} + \frac{2}{15}\right) - \frac{11}{14}$

c. $\frac{3}{4} + \frac{2}{3}$

10. Write the reciprocals of the given numbers.

a. 7

b. 19

c. $\frac{7}{92}$

d. $\frac{6}{8931}$

e. $\frac{8}{21}$

f. $\frac{1}{7}$

11. In the diagram at the right

a. Name three triangles.

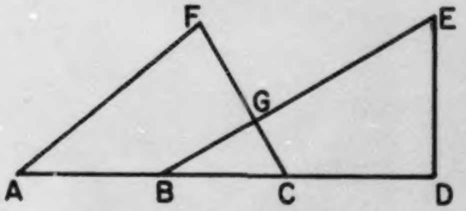
b. What is $\overleftrightarrow{CF} \cap \triangle EBD$?

c. What is $\overline{AF} \cap \overline{BE}$?

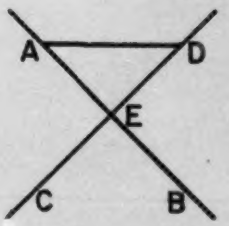
d. What is $\overrightarrow{BG} \cup \overrightarrow{BC}$?

e. Name a point on the B-side of $\overleftrightarrow{FC}$.

f. What is $\overline{AC} \cup \overline{BD}$?



12. Explain why $\overleftrightarrow{AD}$ is in the plane CEB.



Chapter 9

DECIMALS

9-1. Decimal Notation.

In Chapter 2, you learned that a numeral, in base 10, such as 328 can be written as $3(10^2) + 2(10^1) + 8$. This is called the expanded form. In writing a numeral in base 10, in the expanded form, we notice that the exponents of 10 decrease by 1 for each step that we take from left to right. Written as 328, the numeral 328 is said to be written in positional notation. If the base is 10, this is also called decimal notation.

Exercises 9-1a.

1. Write each of the following numerals in expanded form:

    a. 37

    b. 295

    c. 6417

    d. 508

    e. 2069

    f. 3005

    g. 780

    h. 1090

2. Write each of the following numerals in positional notation

    a. $2(10^2) + 5(10) + 9$

    b. $5(10^3) + 4(10^2) + 8(10) + 3$

    c. $9(10^2) + 1(10) + 0$

    d. $6(10^3) + 8(10^2) + 0(10) + 4$

    e. $1(10^3) + 0(10^2) + 0(10) + 7$

3. In connection with the following numerals, indicate the number represented by the underlined digit:

    a. $4\underline{7}8$

    b. $3\underline{2}59$

    c. $6\underline{2}$

    d. $\underline{5}017$

    e. $21\underline{0}3$

    f. $1\underline{5}67$

4. In each case write four-digit numerals whose
    a. tens digit is 3
    b. thousands digit is 7
    c. units digit is 9
    d. hundreds digit is 5

When written in positional form the numeral $5(10^2) + 5(10^1) + 5$ is 555. We notice that the digit in the tens place represents a number which is $\frac{1}{10}$ of the number represented by the digit in the hundreds place. Also, the digit in the units place represents a number which is $\frac{1}{10}$ of the number represented by the digit in the tens place. Thus, when we move a digit one place from left to right the number represented by the digit in its new place is $\frac{1}{10}$ of the number that it represented before.

Let us consider the numeral $4(10^3) + 6(10^2) + 5(10^1) + 9$ and the numeral $4(10^2) + 6(10^1) + 5 + 9\ (?)$. Do you agree that the digit 9 in the second numeral represents a number which is $\frac{1}{10}$ of the number represented by the digit 9 in the first numeral? Therefore, the second numeral may be written as $4(10^2) + 6(10^1) + 5 + (9 \times \frac{1}{10})$.

Let us consider the numeral $7(10^3) + 5(10^2) + 4(10^1) + 3 + 2(?) + 9(?) + 6(?)$. We have seen that the digit that is one place to the right of the units digit represents the value "digit $\times \frac{1}{10}$". Then, the next digit to the right represents the number "digit $\times \frac{1}{10^2}$" and the following digit to the right represents the number "digit $\times \frac{1}{10^3}$". Therefore, the numeral $7(10^3) + 5(10^2) + 4(10^1) + 3 + 2(?) + 9(?) + 6(?)$ is the same as the numeral $7(10^3) + 5(10^2) + 4(10^1) + 3 + 2(\frac{1}{10}) + 9\left(\frac{1}{10^2}\right) + 6\left(\frac{1}{10^3}\right)$.

To express this numeral in positional form we copy the digits in order and write each in its place in order. Thus, we write 7543296.

However, 7543296 is not what we mean since there is no way to tell the whole number places from the fractional places. The punctuation mark used to separate the units place from the fractional places is the decimal point. Thus, the numeral is written 7543.296. This is read, "Seven thousand five hundred forty-three and two hundred ninety-six thousandths" or "Seven five four three point two nine six."

The following chart shows the place values both to the left of and to the right of the units place.

PLACE VALUE CHART

Hundred thousand	Ten thousand	Thousand	Hundred	Ten	Unit	Tenth	Hundredth	Thousandth	Ten-thousandth	Hundred-thousandth
100,000	10,000	1,000	100	10	1	0.1	0.01	0.001	0.0001	0.00001
10^5	10^4	10^3	10^2	10^1	10^0	$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$

EXAMPLE 1: Write $5(10^2) + 7(10^1) + 0(1) + 3(\frac{1}{10})$ in decimal notation (that is, positional notation)
Answer: 570.3

EXAMPLE 2: Write 42.306 in expanded form.
Answer: $4(10^1) + 2(1) + 3(\frac{1}{10}) + 0\left(\frac{1}{10^2}\right) + 6\left(\frac{1}{10^3}\right)$

Exercises 9-1b

1. Write each of the following in decimal notation:

   a. $4(10^3) + 9(10^2) + 6(10) + 7$

   b. $5(10^3) + 6(10^2) + 1(10) + 8 + 3(\frac{1}{10})$

   c. $2(10^2) + 4(10) + 5 + 6(\frac{1}{10}) + 1\left(\frac{1}{10^2}\right)$

   d. $8(10^2) + 0(10^1) + 4 + 3(\frac{1}{10}) + 5\left(\frac{1}{10^2}\right) + 9\left(\frac{1}{10^3}\right)$

   e. $5(10) + 2(\frac{1}{10}) + 4\left(\frac{1}{10^2}\right)$

   f. $4(\frac{1}{10}) + 8\left(\frac{1}{10^2}\right) + 3\left(\frac{1}{10^3}\right)$

   g. $3(10^2) + 7(10) + 2 + 0(\frac{1}{10}) + 6\left(\frac{1}{10^2}\right)$

   h. $2\left(\frac{1}{10^2}\right) + 0\left(\frac{1}{10^3}\right) + 6\left(\frac{1}{10^4}\right)$

   i. $3 + 5(\frac{1}{10}) + 0\left(\frac{1}{10^2}\right) + 7\left(\frac{1}{10^3}\right)$

2. Write each of the following in expanded form:

   a. 7862

   b. 437.9

   c. 28.64

   d. 347.15

   e. 96.372

   f. 2.465

   g. .384

   h. .013

   i. 24.09

   j. .0039

3. Write each of the following in words:

   a. 658

   b. 3.2

   c. 4.73

   d. 58.29

   e. 759.6

   f. 48.07

   g. 3.209

   h. 37.0106

4. Write in decimal form:
   a. Five and fifty-two hundredths
   b. Seven hundred sixty-two and nine-tenths
   c. Three hundred and fifty-two hundredths
   d. Fourteen hundredths
   e. Two and seven thousandths
   f. Sixty and seven hundredths

*5. Write 0.1_{two} in base 10.

*6. Write $\frac{1}{2}$ in the duodecimal system.

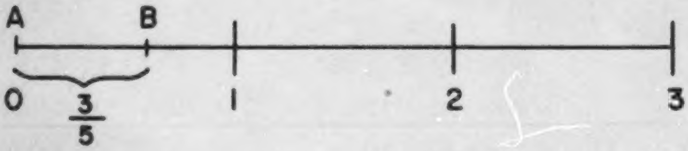
*7. Write $\frac{5}{6}$ in the duodecimal system.

*8. Change 10.011_{two} to base 10.

Expressing a Rational Number in Decimal Form

In Chapter 6, a rational number was defined as the quotient of a whole number by a counting number. This quotient was interpreted on the number line.

For example, the rational number $\frac{3}{5}$ was interpreted as follows: Divide a line segment 3 units long into 5 equal parts. The measure of the length of one of the parts is $\frac{3}{5}$. The procedure is illustrated below.



Suppose we want to express $\frac{3}{5}$ in decimal form. We know that $\frac{3}{5}$ is less than 1, so each of the digits to the left of the decimal point is zero. We need to find a digit for each place to the right of the decimal point. That is, we need to find the digits indicated by question marks in the following number sentence.

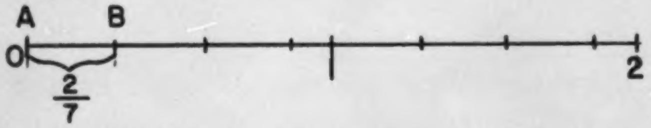
$$\frac{3}{5} = ?\left(\frac{1}{10}\right) + ?\left(\frac{1}{10^2}\right) + ?\left(\frac{1}{10^3}\right) + \ldots$$

First, let us look for the digit in the tenths place. We want to find the length, in tenths, of the segment labeled $\overline{AB}$ in the figure. A length of 3 units is 30 tenths of a unit. We have divided the length of 3 units into 5 equal parts. Then each part must be $\frac{30}{5}$ tenths of a unit in length. But $\frac{30}{5} = 6$, so the length of $\overline{AB}$, in tenths of a unit, is exactly 6.

Thus, $\frac{3}{5}$ in decimal form, is equal to .6 .

Suppose we wish to express the rational number $\frac{2}{7}$ in decimal form. The rational number $\frac{2}{7}$ is interpreted as follows:

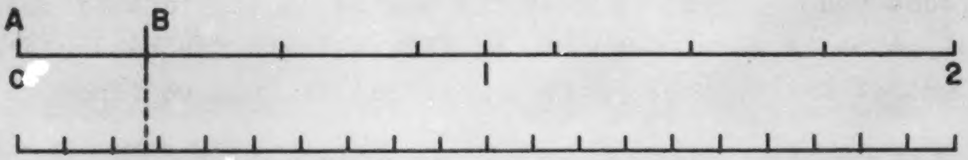
Divide a line segment 2 units long into 7 equal parts. The measure of the length of one of the parts is $\frac{2}{7}$. The method is shown below.



We know that $\frac{2}{7}$ is less than 1. Therefore, each of the digits to the left of the decimal point is zero. We need to find a digit for each place to the right of the decimal point. That is, we need to find the digits indicated by question marks in the following number sentence:

$$\frac{2}{7} = ?\left(\frac{1}{10}\right) + ?\left(\frac{1}{10^2}\right) + ?\left(\frac{1}{10^3}\right) + \ldots.$$

Let us take them in order. First, we will find the digit in the tenths place. We want to find the length, in tenths, of the segment labeled $\overline{AB}$ in the figure.



Since a length of 2 units is 20 tenths of a unit, and we have divided the length of 2 units into 7 equal parts, each part must be $\frac{20}{7}$ tenths of a unit in length. But $\frac{20}{7} = 2\frac{6}{7}$, so the length of $\overline{AB}$, in tenths of a unit, is between 2 and 3. It is 2 tenths of a unit and some left over. The tenths digit is "2".

Now let us find the hundredths digit. A length of 2 units is 200 hundredths of a unit. Since we have divided the length of 2 units into 7 equal parts, each part must be $\frac{200}{7}$ hundredths of a unit in length.

$$\frac{200}{7} = 28\frac{4}{7}$$

Therefore, the length of $\overline{AB}$, in hundredths of a unit lies between 28 and 29. It is 2 tenths of a unit plus 8 hundredths of a unit and some left over. The hundredths digit is "8".

Do you see how we can find the thousandths digit. Notice that 2 units is 2000 thousandths of a unit. Therefore, the length of AB is $\frac{2000}{7}$ thousandths of a unit.

$$\frac{2000}{7} = 285\frac{5}{7}$$

Therefore, the length of $\overline{AB}$, in thousandths of a unit, is between 285 and 286. It is .2 + .08 + .005 + some left over.

The steps above can be done by division as follows:

$$\begin{array}{r} 0 \\ 7\overline{)2} \\ \underline{0} \\ 2 \end{array}$$

The whole number part of the quotient is zero. Therefore, each digit to the left of the decimal point is a zero.

$$\begin{array}{r} 0.2 \\ 7\overline{)2.0} \\ \underline{1\ 4} \\ 6 \end{array}$$

$\frac{2}{7}$ is 2 tenths and some left over.

$$\begin{array}{r} 0.28 \\ 7\overline{)2.00} \\ \underline{1\ 4} \\ 60 \\ \underline{56} \\ 4 \end{array}$$

$\frac{2}{7}$ is 28 hundredths and some left over.

$$\begin{array}{r} 0.285 \\ 7\overline{)2.000} \\ \underline{1\ 4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 5 \end{array}$$

$\frac{2}{7}$ is 285 thousandths and some left over.

Continuing the division we have Enough steps have been shown to indicate that $\frac{2}{7}$ could never be written as an exact decimal. In fact, if we continued the division we would find that the digits 285714 would be repeated over and over again. Can you explain why?

$$\begin{array}{r} 0.285714 \\ 7\overline{)2.000000} \\ \underline{1\ 4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 2 \end{array}$$

You should do this type of problem by short division.

$$\begin{array}{l} \underline{0.285714}.... \\ 7\ 2.000000 \end{array}$$

Exercises 9-1c

1. Write each of the following as decimals.

a. $\frac{3}{10}$

b. $\frac{3}{2}$

c. $\frac{7}{4}$

d. $\frac{1}{8}$

e. $\frac{15}{8}$

f. $\frac{100}{16}$

2. Write each of the following as decimals to the hundredths place:

a. $\frac{2}{3}$

b. $\frac{5}{9}$

c. $\frac{5}{7}$

d. $1\frac{11}{12}$

e. $\frac{1}{6}$

f. $\frac{4}{15}$

g. $\frac{7}{13}$

h. $\frac{19}{17}$

i. $\frac{3}{14}$

3. Change each number of the pair to a decimal to the hundredths place and write them so that the larger number is first.

a. $\frac{3}{7}$; $\frac{17}{42}$

b. $\frac{5}{16}$; $\frac{25}{83}$

c. $\frac{7}{12}$; $\frac{23}{42}$

9-2. Arithmetic Operations With Decimals.

Decimals are found to be very useful in industry and in science. In order to be able to use decimals in solving problems we must learn how to add, subtract, multiply, and divide with decimals.

Suppose we wish to add two decimals, for example, $0.73 + 0.84$. We know how to add these numbers without the decimal places: $73 + 84 = 157$. How do we handle the decimal places?

We may proceed as follows:

$$0.73 = 73 \times \frac{1}{100} \quad \text{and} \quad 0.84 = 84 \times \frac{1}{100}$$

Therefore,

$$\begin{aligned} 0.73 + 0.84 &= (73 \times \frac{1}{100}) + (84 \times \frac{1}{100}) \\ &= (73 + 84) \times \frac{1}{100} \\ &= 157 \times \frac{1}{100} = 1.57 \end{aligned}$$

Notice that we have used the distributive property here.

Suppose that we wish to add a two decimal place number and a three decimal place number, for example, $0.73 + 0.125$. Just as before we first rewrite our numbers as fractions:

$$0.73 = 73 \times \frac{1}{100} = 730 \times \frac{1}{1000} \quad \text{and} \quad 0.125 = 125 \times \frac{1}{1000}.$$

We write $73 \times \frac{1}{100}$ as $730 \times \frac{1}{1000}$, since then the factor $\frac{1}{1000}$ will appear in both numbers.

$$\begin{aligned} 0.73 + 0.125 &= (730 \times \frac{1}{1000}) + (125 \times \frac{1}{1000}) \\ &= (730 + 125) \times \frac{1}{1000} \\ & 855 \times \frac{1}{1000} = 0.855 \end{aligned}$$

These examples can be handled more conveniently by writing one number below the other as follows:

$$\begin{array}{r} .73 \\ + \underline{\ \ .84} \\ 1.57 \end{array} \qquad\qquad \begin{array}{r} 0.73 \\ + \underline{0.125} \\ 0.855 \end{array}$$

Notice that we write the decimal points directly under one another. This is because we want to add the digits in the $\frac{1}{10}$ place in the first number to the digit in the $\frac{1}{10}$ place in the second, and the digit in the $\frac{1}{100}$ place in the first number to the digit in the $\frac{1}{100}$ place in the second, etc. Thus,

$$0.73 = \frac{7}{10} + \frac{3}{100} \quad \text{and} \quad 0.84 = \frac{8}{10} + \frac{4}{100}$$

and therefore

$$\begin{aligned} 0.73 + 0.84 &= \frac{7+8}{10} + \frac{3+4}{100} \\ &= \frac{15}{10} + \frac{7}{100} \\ &= 1.57 \end{aligned}$$

Subtraction can be handled in the same way. For example,

$$\begin{array}{r} 0.84 \\ -\ \underline{0.73} \\ 0.11 \end{array} \qquad\qquad \begin{array}{r} 0.83 \\ -\ \underline{0.74} \\ .09 \end{array}$$

Exercises 9-2a

1. Add the following numbers.

   a. .3 + .5

   b. 0.73 + 0.59

   c. .6 + 0.85

   d. 0.719 + 0.382

   e. 1.002 + 0.00102

   f. 1.05 + 0.75 + 21.5

   g. 23.04 + 9.6 + 16.58

2. Subtract the following numbers.

   a. .9 - .3

   b. .84 - .76

   c. .35 - .09

   d. 1.36 - .97

   e. 0.625 - 0.550

   f. 0.500 - 0.125

   g. 1.005 - 0.0005

3. Perform the operations indicated

   a. $2.05 + 3.48 - 1.72$

   b. $8.675 + 4.837 - 0.066$

   c. $5.073 - 1.902 + 2.463$

   d. $10.105 - 3.769 + 5.909$

   e. $0.4075 - 0.32 + 0.076$

   f. $8.035 - 2.796 - 0.308$

4. There are 16 ounces in 1 pound. Which is heavier, 7 ounces or 0.45 lb.?

5. In a mile relay race, four men each run a quarter-mile. Their times are 48.3 seconds, 47.9 seconds, 49.0 seconds, and 48.5 seconds. The meet record is 3 minutes 8.5 seconds. By how many seconds did this team fail to break the record?

6. The rainfall in a certain city for a year was 30.04 inches. The rainfall record by months was as follows:

   January - 1.39 inches
   February - 0.92 inches
   March - 3.14 inches
   April - 4.86 inches
   May - 3.49 inches
   June - 1.97 inches

   How much rain fell during the rest of the year?

*7. Add $10.01_{two} + 1.01_{two}$ and then express the answer in the base 10.
   Hint: In the base two: $0.1_{two} = \frac{1}{2}$ and $0.01_{two} = \frac{1}{4}$

Suppose we wish to multiply two decimals, for example, 0.3×0.25. We know how to multiply these numbers without the decimal places: $3 \times 25 = 75$. Just as before, we write

$$\begin{aligned} 0.3 \times 0.25 &= 3 \times \frac{1}{10} \times 25 \times \frac{1}{100} \\ &= 3 \times 25 \times \frac{1}{10} \times \frac{1}{100} \\ &= 75 \times \frac{1}{1000} \\ &= 0.075 \end{aligned}$$

1. How many digits are there to the right of the decimal point in 0.3?
2. How many digits are there to the right of the decimal point in 0.25?
3. What is the sum of the answers to (1) and (2)?
4. How many digits are there to the right of the decimal point in 0.075?
5. Compare the answers to (3) and (4).

Now multiply 0.42×0.29. What is your answer? Answer the five questions above, (1), (2), (3), (4), (5), for these numbers. Do the answers to (3) and (4) still agree?

To find the number of decimal places when two numbers are multiplied, add the number of decimal places in the two numbers.

For example, suppose we wish to multiply .732 by .25. The first number has three decimal places and the second has two, so there will be five places in the answer. We multiply 732×25, and then mark off five decimal places in the answer, counting from right to left.

$$\begin{array}{r} .732 \\ \underline{.25} \\ 3660 \\ \underline{1464} \\ .18300 \end{array}$$

Thus $.732 \times .25 = .18300$

The next operation we will consider is division. For example, how do we divide .125 by .5 .

$$\frac{.125}{.5} = \frac{\frac{125}{1000}}{\frac{5}{10}}$$

$$= \frac{125}{1000} \times \frac{10}{5}$$

$$= \frac{125}{5} \times \frac{10}{1000}$$

$$= 25 \times \frac{1}{100}$$

$$= .25$$

Let us try another example, $.5 \div .04$

We shall use the fact that $5 \div 4 = 1.25$. You were shown how to do this in the last section.

$$\frac{.5}{.04} = \frac{\frac{5}{10}}{\frac{4}{100}}$$

$$= \frac{5}{10} \times \frac{100}{4}$$

$$= \frac{5}{4} \times \frac{100}{10}$$

$$= 1.25 \times 10 = 12.5$$

We will now try an example with a repeating decimal. In the last section it was shown that $\frac{2}{7} = 0.285714285714 \ldots$ where 285714 is repeated over and over again, without end.

Let us try the problem $.02 \div .7$.

$$\frac{.02}{.7} = \frac{\frac{2}{100}}{\frac{7}{10}}$$

$$= \frac{2}{100} \times \frac{10}{7}$$

$$= \frac{2}{7} \times \frac{10}{100}$$

$$= .285714 \ldots\ldots \times \frac{1}{10}$$

$$= .0285714 \ldots\ldots$$

In Chapter 6 we learned the following property-

If the numerator and denominator of a fraction are both multiplied by the same number (not zero), the number which the fraction represents is unchanged.

Thus, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$. We shall use this property for fractions with decimals. For example,

$$\frac{.125}{.5} = \frac{1.25}{5} \quad \text{and} \quad \frac{.02}{.7} = \frac{.2}{7}$$

In these cases, we have multiplied the numerator and denominator of each fraction by 10.

Another example is

$$\frac{.5}{.04} = \frac{50}{4}$$

By what number were numerator and denominator multiplied in this example?

In dividing one decimal by another our first step is to shift all digits in the divisor to the left of the decimal point by multiplying dividend and divisor by a suitable power of ten.

Thus:

$$.125 \div .5 = \frac{.125}{.5} = \frac{1.25}{5}$$

$$.5 \div .04 = \frac{.5}{.04} = \frac{50}{4}$$

$$.02 \div .7 = \frac{.02}{.7} = \frac{.2}{7}$$

After all digits in the divisor have been shifted to the left of the decimal point, the division is performed.

$$\begin{array}{r} .25\\ 5\overline{)1.25}\\ \underline{1\,0}\\ 25\\ \underline{25}\\ 0 \end{array} \qquad \begin{array}{r} 12.5\\ 4\overline{)50.0}\\ \underline{4}\\ 10\\ \underline{8}\\ 20\\ \underline{20}\\ 0 \end{array} \qquad \begin{array}{r} .0285\\ 7\overline{).2000}\\ \underline{14}\\ 60\\ \underline{56}\\ 40\\ \underline{35}\\ 5 \end{array}$$

These three examples should all be done by short division:

$$\begin{array}{r} .25\\ 5\overline{)1.25} \end{array} \qquad \begin{array}{r} 12.5\\ 4\overline{)50.0} \end{array} \qquad \begin{array}{r} .0285\\ 7\overline{).2000} \end{array}$$

Notice that the decimal point in the answer is directly above the decimal point in the dividend.

Exercises 9-2b

(Assume that the decimals are exact.)

1. Find the following products.

a.	$.8 \times .7$	d.	$.0025 \times 2.5$
b.	$.06 \times .9$	e.	1.2×120
c.	$.009 \times .09$	f.	$.135 \times .202$

2. Find the following quotients.

a. $2.505 \div 5$
b. $.06528 \div 8$
c. $.009 \div 30$
d. $.013596 \div .3$
e. $.015 \div .05$
f. $.574 \div .4$
g. $2.04 \div .008$
h. $6.31845 \div .009$

3. Express the following fractions as decimals.

a. $\frac{3000}{8}$
b. $\frac{300}{8}$
c. $\frac{30}{8}$
d. $\frac{3}{8}$
e. $\frac{3}{80}$
f. $\frac{3}{800}$

4. $\frac{.015 \times .0025 \times 2.5}{.05 \times .03} = ?$

5. $\frac{.62 \times 1.44 \times .072}{.03 \times .04} = ?$

6. The Lincoln School has an enrollment of 325 students. On a stormy day .88 of the students attended school. How many students attended school that day?

7. Mr. Brown drives 8.6 miles each day on his way to work and return. How many miles will he drive during a month which has 19 working days?

*8. Find the following product of two base seven numbers

$$1.14_{\text{seven}} \times 2.4_{\text{seven}} = 2 + \frac{4}{7}$$

Hint: $1.14_{\text{seven}} = 1 + \frac{1}{7} + \frac{4}{7^2}$ and $2.4_{\text{seven}} = 2 + \frac{4}{7}$

9-3. Decimal Expansion.

We have learned how to name a rational number by a decimal numeral (or decimal). Let us look more closely at rational numbers in decimal form. You recall that $\frac{1}{8}$ is the quotient of

1 divided by 8.

$$\begin{array}{rl} 0.1250\ldots & \\ 8\overline{)1.000} & \\ \underline{8} & \\ 20 & \longrightarrow \text{After the first subtraction the remainder is 2} \\ \underline{16} & \\ 40 & \longrightarrow \text{The second remainder is 4} \\ \underline{40} & \\ 0 & \longrightarrow \text{The third remainder is 0} \\ \underline{0} & \\ 0 & \longrightarrow \text{The fourth remainder is 0} \end{array}$$

We usually stop our division process when we obtain a zero remainder. You can see that once we obtain a zero remainder we will continue to get zero remainders if we keep on dividing. Also, all digits in the quotient after we obtain a zero remainder will also be zero.

$\frac{1}{8} = 0.125000\ldots$ (The dots indicate that the decimal numeral never ends.)

Therefore, 0.125000... is a repeating decimal with 0 repeating over and over again.

Consider the rational number $\frac{1}{3}$. We can easily find the decimal numeral which names $\frac{1}{3}$ by dividing 1 by 3.

$$\begin{array}{r} 0.333333\ldots \\ 3\overline{)1.000000} \end{array}$$

Without performing the division do you know what digits will appear in each of the next 6 places? Will we obtain a remainder of zero if we continue the division? Do you agree that 0.333.. is a repeating decimal with 3 repeating over and over again?

This naming of a rational number by a decimal numeral in which there is a digit for every decimal place is called decimal expansion. The decimal expansions of $\frac{1}{8}$ and $\frac{1}{3}$ seem to indicate that every rational number can be named by a decimal expansion.

Let us look into this matter further.

Class Discussion Exercises

Let us look at the decimal which names the rational number $\frac{1}{7}$.

$$\begin{array}{r} 0.142857142857\ \ldots \\ 7\overline{)1.000000000000} \end{array}$$

1. Can you tell, without performing the division, the digits that should appear in the next 6 places?
2. Is there a block of digits which repeat endlessly? Let us place a horizontal bar over the block of digits which repeats. Thus, $0.142857\overline{142857}\ldots$
3. Name $\frac{1}{11}$ by a decimal numeral.
4. How soon can you recognize the pattern of repetition?
5. Will there be a zero remainder if you keep on dividing?
6. Does the decimal name for $\frac{1}{11}$ repeat? How will you indicate this?
7. Do you notice that the decimal name for $\frac{1}{11}$ repeats as the remainder repeats?
8. Let us check the idea in Step 7 by finding the decimal name for $\frac{1}{7}$.

$$\begin{array}{l} \quad 0.1428571\ \ldots. \\ 7\overline{)1.0000000} \\ \quad \underline{7} \\ \quad 30 \longrightarrow \text{First remainder is } 3 \\ \quad \underline{28} \\ \quad\ 20 \longrightarrow \text{Second remainder is } 2 \\ \quad\ \underline{14} \\ \quad\ \ 60 \longrightarrow \text{Third remainder is } 6 \\ \quad\ \ \underline{56} \\ \quad\ \ \ 40 \longrightarrow \text{Fourth remainder is } 4 \\ \quad\ \ \ \underline{35} \\ \quad\ \ \ \ 50 \longrightarrow \text{Fifth remainder is } 5 \\ \quad\ \ \ \ \underline{49} \\ \quad\ \ \ \ \ 10 \longrightarrow \text{Sixth remainder is } 1 \\ \quad\ \ \ \ \ \ \underline{7} \\ \quad\ \ \ \ \ \ 3 \longrightarrow \text{Seventh remainder is } 3 \end{array}$$

Note that the seventh remainder is the first time that the remainder repeats. Is the seventh digit to the right of the decimal point the same as the first?

If you continue the division will the eighth remainder be the same as the second remainder? Will the eighth digit to the right of the decimal point be the same as the second digit to the right of the decimal point?

9. Check this idea again by dividing 1 by 37 to find a decimal numeral naming $\frac{1}{37}$.

Do you agree with the following conclusion? Every rational number can be named by a decimal numeral which either repeats a single digit or a block of digits over and over again.

Exercises 9-3a

1. Write a decimal numeral for $\frac{1}{13}$.
   a. How soon can you recognize a pattern?
   b. Does this decimal numeral end?
   c. How should you indicate that it does not end?
   d. Is there a digit or set of digits which repeats periodically?
   e. How should you indicate this?

2. Write decimal numerals for:

   a. $\frac{1}{3}$ b. $\frac{1}{6}$ c. $\frac{1}{9}$ d. $\frac{4}{11}$ e. $\frac{7}{15}$

   How soon can you recognize a pattern in each case? In each case locate the step in which the remainders begin to repeat. Do the digits in the decimal numeral begin to repeat at this step?

3. Write decimal numerals for

a. $\frac{1}{11}$

b. $\frac{2}{11}$

c. $\frac{3}{11}$

d. $\frac{9}{11}$

e. $\frac{14}{11}$

f. $\frac{23}{11}$

Study these decimal numerals and see if you can find a relationship.

1. between the decimal numeral naming $\frac{1}{11}$ and the decimal numeral naming $\frac{2}{11}$
2. between the decimal numeral naming $\frac{1}{11}$ and the decimal numerals naming $\frac{3}{11}$, $\frac{9}{11}$, $\frac{14}{11}$, $\frac{23}{11}$.
3. Can you find a decimal numeral for $\frac{5}{11}$ without dividing?
4. Can you find a decimal numeral for $\frac{7}{11}$ without dividing?
5. Is it true that the decimal numeral for $\frac{3}{11}$ is three times that of $\frac{1}{11}$?

Exercises 9-3b

1. For each of the following numbers:

Find a decimal numeral.

Indicate the repetition of a digit or a set of digits by placing a horizontal bar over the digit or block of digits which repeat.

a. $\frac{1}{2}$

b. $\frac{1}{5}$

c. $\frac{3}{4}$

d. $\frac{3}{5}$

e. $\frac{7}{8}$

f. $\frac{4}{5}$

g. $\frac{2}{3}$

h. $\frac{9}{11}$

i. $\frac{5}{9}$

j. $\frac{11}{15}$

2. Find decimal numerals for the first number in each group and calculate the others without dividing.

a. $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}$

b. $\frac{1}{10}, \frac{4}{10}, \frac{7}{10}$

c. $\frac{1}{8}, \frac{3}{8}, \frac{7}{8}$

d. $\frac{1}{11}, \frac{6}{11}, \frac{9}{11}$

e. $\frac{1}{20}, \frac{3}{20}, \frac{11}{20}$

f. $\frac{1}{50}, \frac{42}{50}, \frac{47}{50}$

g. $\frac{1}{1000}, \frac{112}{1000}, \frac{927}{1000}$

9-4. Rounding Decimal Numerals.

The length of the Missouri River is 2,466 miles. In talking about the length of the Missouri River a boy said that it is 2,500 miles. Would you say that he gave the length of the river incorrectly? Actually, the boy gave the length of the river to the nearest hundred miles. We say that the length of the river was rounded to the nearest hundred miles.

We often use rounded numbers because they are easier to remember and easier to work with. In many cases, the rounded number is just as useful as the precise number.

The population of Colorado is 1,625,089. This may be rounded to 1,600,000. Do you agree that we rounded to the nearest hundred thousand in this case? In talking about the population of states the rounded number would be more convenient than the exact number for most uses.

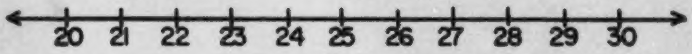
The height of Mt. Rainier is 14,410 feet. This may be rounded to 14,400 feet. Do you agree that we rounded to the nearest hundred feet in this case?

When we round a number we give an approximation of it. Often, an approximation is all we need. We may prefer the approximation because it is easier to remember and to use. As you would expect, we choose an approximation close to the actual number.

Example 1:

We say that 23, rounded to the nearest 10 units, is 20. Why don't we say 30?

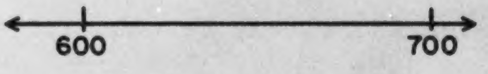
Look at the number line and find 23. Is it closer to 20 or to 30? Locate the point halfway between 20 and 30. On which side of this halfway point is 23?



Example 2:

Round 677 to the nearest hundred.

Touch with your finger the point where you think 677 is located. What is the number for the point halfway between 600 and 700? On which side of this halfway point is 677?



Did you decide that 677, rounded to the nearest 100, is 700?

Example 3:

Round 450 to the nearest hundred.

In this case, 450 is exactly halfway between 400 and 500. In such a case, we agree to round the number to the higher hundred. Thus, 450 rounded to the nearest hundred, is 500.

Exercises 9-4a

1. Round these numbers to the nearest hundred:

a. 280
b. 314
c. 1,436
d. 850
e. 749
f. 3,555
g. 439
h. 978
i. 5,250

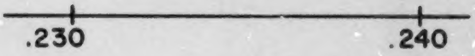
2. Round these numbers to the nearest thousand:

a. 5,820
b. 3,399
c. 2,949
d. 14,550
e. 62,853
f. 144,499
g. 144,501
h. 144,500
i. 326,495

3. Round these numbers to the nearest ten thousand:

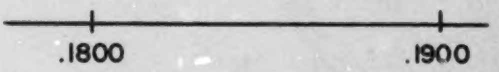
a. 734,159
b. 159,023
c. 72,400
d. 195,682
e. 568,431
f. 85,000

Look at the portion of the number line at the right. What number is halfway between .230 and .240? Where is .238? Is it closer to .230 or to .240? How would you round .238 to the nearest hundredth?



Example 1:

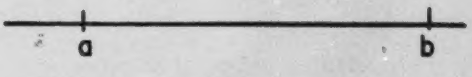
Round .1839 to two decimal places.



Touch with your finger the place where you think .1839 is on the number line. What is halfway between .1800 and .1900? On which side of the halfway point do you place .1839? Would you round off .1839 to .1800 or to .1900?

Example 2:

Round .2673 to two decimal places.



Give values to a and b of the form .??00 so that a is the closest number to .2673 on the left and b is the closest number to .2673 on the right. Locate .2673 between the values for a and b. Which is closer to .2673, a or b? Did you decide that .2673 rounded to two decimal places is .27?

Exercises 9-4b

1. Explain why each of the following is correct:

	Number	Rounded to two Decimal Places
a.	0.237	0.24
b.	0.241	0.24
c.	0.244	0.24
d.	0.1949	0.19

2. Round each of the following to two decimal places:

a. 48.362
b. 0.518
c. 35.016
d. 6.012
e. 0.0053
f. 0.097

3. Round each of the following to one decimal place:

a. 16.38
b. 48.72
c. 108.05
d. 0.074
e. 0.051
f. 1.16
g. 3.1415
h. 68.07
i. 43.03

4. Round each of the following to three decimal places:

a. 4.0486
b. 17.1074
c. 0.0006
d. 0.00159
e. 185.7308
f. 62.9119

5. Write each of the following as a decimal. Round the result to one decimal place.

a. $\frac{62}{100}$
b. $\frac{1}{3}$
c. $\frac{4}{9}$
d. $\frac{5}{6}$
e. $2\frac{1}{11}$
f. $3\frac{4}{15}$

6. Write each of the following as a decimal. Round the result to two decimal places:

a. $\frac{531}{1000}$
b. $\frac{2}{3}$
c. $\frac{5}{9}$
d. $1\frac{7}{12}$
e. $\frac{5}{7}$
f. $\frac{1}{6}$

9-5. Summary.

A number may be written in decimal notation or in expanded form. For example,

Decimal Notation	Expanded Form
489	$4(10^2) + 8(10^1) + 9$
3,056	$3(10^3) + 0(10^2) + 5(10^1) + 6$

When one reads a number, the decimal point is read "and" or "point".

A rational number may be named by a decimal by dividing the numerator of the rational number by the denominator of the rational number. If the division is continued we discover that we reach a remainder of zero or that, after a certain point, the remainders begin to repeat. After we reach a remainder of zero the succeeding digits in the quotient are zero. After we reach the first repeating remainder the succeeding digits in the quotient are repetitions of the preceding digits in the quotient in the same order. If we continue the division the digits in the quotient will repeat in blocks over and over again.

Thus, each rational number may be named by a repeating decimal. In some cases, the repeating digits will be zeros. In other cases, the digits will repeat in blocks.

The following are important rules for performing arithmetic operations with decimals.

1. In the addition or subtraction of numbers containing decimals, write the numbers in vertical order so that the decimal points are directly under one another. Then add or subtract as with whole numbers. The decimal point in the answer is directly under the other decimal points.
2. To find the number of decimal places when two numbers are multiplied, add the number of decimal places in the two numbers.

3. In dividing one decimal by another shift all digits in the divisor to the left of the decimal point by multiplying dividend and divisor by a suitable power of 10. Then divide as with whole numbers. The decimal point in the answer is directly above the decimal point in the dividend.

In rounding decimal numerals use the following rule:

Note the digit which is one place to the right of the number of places to which you are rounding. If this digit is 5 or more then increase the preceding digit by 1. If this digit is less than 5 drop it.

9-6. Chapter Review.

Exercises 9-6.

1. Write each of the following in expanded form:

   a. 79 c. 6,028

   b. 453 d. 5,739

2. Write each of the following in positional notation:

   a. $3(10^2) + 5(10^1) + 2$

   b. $4(10^3) + 9(10^2) + 3(10^1) + 7$

   c. $8(10^3) + 9(10^1) + 4$

3. Write each of the following in words:

   a. 54.3 b. 169.05

4. Write in decimal form:

   a. Six and nine-tenths

   b. Ninety and six hundredths

5. Write each of the following as decimals to the nearest hundredth:

   a. $\frac{3}{10}$ c. $1\frac{3}{8}$ e. $2\frac{3}{4}$

   b. $\frac{5}{6}$ d. $\frac{2}{9}$ f. $\frac{2}{3}$

6. Change each number of the pair to a decimal to the hundredths place and write them so that the larger number is first.

   a. $\frac{11}{25}$, $\frac{7}{15}$

   b. $\frac{4}{11}$, $\frac{7}{20}$

   c. $\frac{17}{53}$, $\frac{5}{18}$

7. Add the following numbers.

   a. 0.58 + 2.97

   b. 1.602 + 3.059

8. Subtract the following numbers.

   a. 0.93 - 0.48

   b. 3.057 - 1.928

9. Perform the operations indicated

   a. 3.09 + 4.16 - 2.07

   b. 7.821 - 2.056 - 1.309

10. Find the following products.

   a. .35 × .09

   b. .008 × .7

11. Find the following quotients.

   a. .05691 ÷ .3

   b. 1.2586 ÷ .007

12. Write decimal numerals for the following

a. $\frac{3}{11}$ b. $\frac{7}{9}$ c. $\frac{5}{13}$

How soon can you recognize a pattern in each case?

In each case, locate the step in which the remainders begin to repeat. Do the digits in the decimal numeral begin to repeat at this step?

13. Round these numbers to the nearest hundred:

a. 375 c. 938

b. 1250 d. 12,549

14. Round these numbers to the nearest thousand:

a. 6,712 c. 76,802

b. 485,350 d. 5,349

15. Round each of the following to one decimal place:

a. 2.47 c. 96.55

b. 385.063 d. 1,043.142

16. Round each of the following to two decimal places:

a. 3.267 c. 0.0172

b. 489.059 d. 5,829.135

17. Round each of the following to three decimal places:

a. 25.0947 c. 47.69542

b. 3.125639 d. 78.04375

18. Write each of the following as a decimal. Round the result to two decimal places:

a. $\frac{7}{8}$ b. $\frac{4}{17}$

9-7. Cumulative Review.

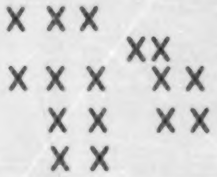
Exercises 9-7

How Are You Doing?

1. What property is illustrated below?

$$5 \cdot 3 + 5 \cdot 2 = 5(3 + 2)$$

2. The number of x's at the right is written in numerals in four different bases. Which numerals are correct?



a. 100_{four} c. 16_{ten}

b. 14_{twelve} d. 31_{five}

3. Write 7^4 without using exponents. What is its value?

4. Answer true or false.

   a. Every number can be completely factored in only one way.

   b. Zero is a factor of every number.

   c. It has been proved that every even prime number is the sum of two prime numbers.

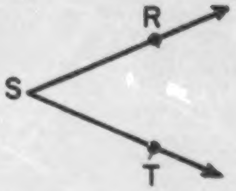
   d. The number one is a factor of every number.

   e. The intersection set of the set of prime numbers and the set of even numbers is the empty set.

   f. If a, b, c, and d are whole numbers and b and d are not equal to zero, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

5. In the diagram at the right

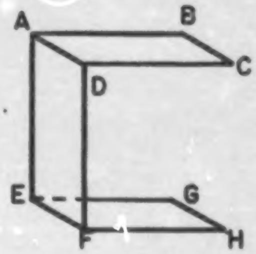


a. What is $\overrightarrow{SR} \cup \overrightarrow{ST}$?

b. What is point S called?

c. What is T-side $\overleftrightarrow{SR} \cap$ R-side $\overleftrightarrow{ST}$?

6. In the figure on the right



a. What is plane ABC $\cap$ plane EFD?

b. Name two skew lines.

c. What is $\overline{AD} \cap \overline{BC}$?

d. What is $\overrightarrow{FE} \cup \overrightarrow{FH}$?

7. Perform the indicated operations and simplify.

a. $1\frac{3}{5} \times 2\frac{1}{2}$

b. $\frac{4}{5} \times 4\frac{1}{2}$

c. $1\frac{2}{3} \times 2\frac{1}{10}$

8. Write each of the following as decimals.

a. $\frac{17}{8}$ c. $\frac{9}{10}$ e. $1\frac{3}{4}$

b. $\frac{21}{4}$ d. $\frac{49}{56}$

9. Mr. Miller is a salesman for the General Products Corporation. He is permitted an allowance of 8.5 cents per mile for car travel. What is his allowance for a trip of 634 miles?

10. Divide, then use the inverse operation for checking.

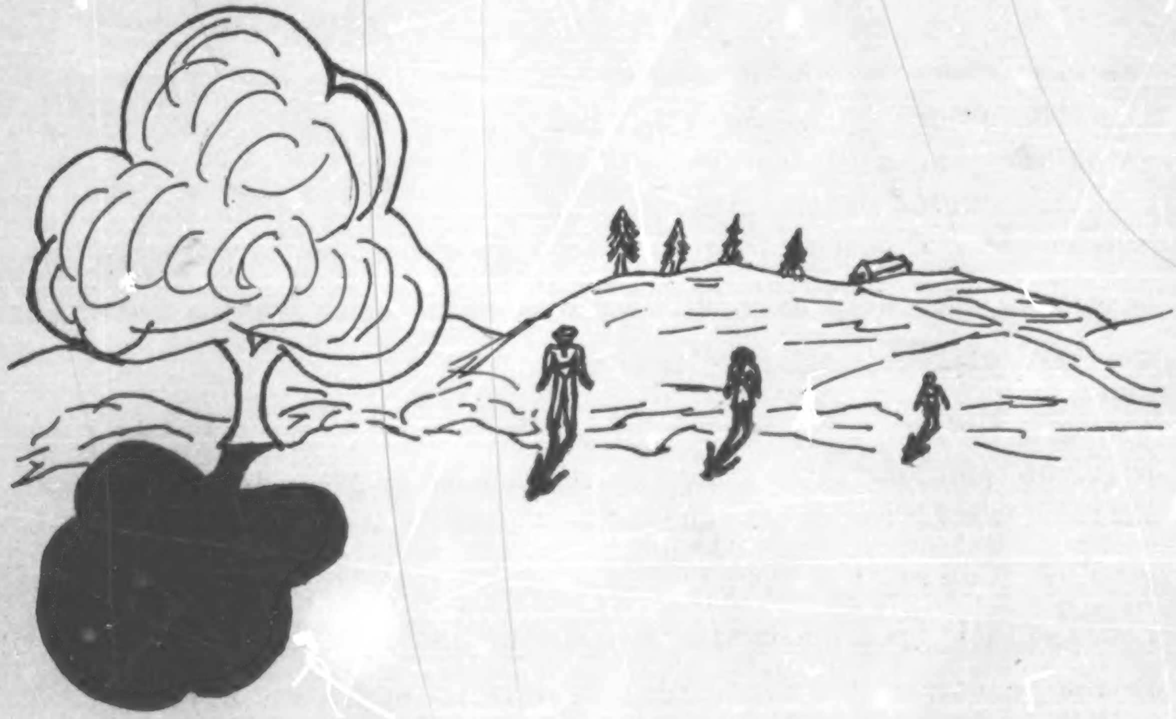
$28\overline{)84868}$

Chapter 10

RATIO AND PERCENT

10-1. Ratio

One sunny day Tony measured the length of the shadow made by each member of his family. He also measured the length of the shadow made by a big tree in the yard. He found that his father, who was 72 inches tall, had a shadow 48 inches long. His mother, who was 63 inches tall, had a shadow 42 inches long. His little brother, who was only 30 inches high, had a shadow 20 inches long. He didn't know how tall the tree was, but its shadow was 40 feet long.



He wrote this information in a table.

	Shadow length	Height
Father	48 inches	72 inches
Mother	42 inches	63 inches
Brother	20 inches	30 inches
Tree	40 feet	?

Tony saw that the taller people had longer shadows. He wondered if he could find a useful way to compare the numbers, and tried division. He divided the shadow length of his little brother by the number of inches in his brother's height.

$$\frac{20}{30} = \frac{2 \cdot 10}{3 \cdot 10} = \frac{2}{3}$$

He tried the same thing with his father's shadow length and height.

$$\frac{48}{72} = \frac{2 \cdot 24}{3 \cdot 24} = \frac{2}{3}$$

Do this for the mother. Do you get $\frac{2}{3}$?

These heights and shadows were all measured at the same time. The number $\frac{2}{3}$ holds for all of them. This number, $\frac{2}{3}$, holds for the tree, too.

The shadow of the tree was measured to be 40 feet. Can you guess the height of the tree?

The height of the tree must be 60 feet in order that the measure of its shadow length divided by the number of feet in its height may be $\frac{2}{3}$. With this reasoning you can find the height of the tree without measuring it.

In Chapter 6 you worked with numerals like $\frac{2}{3}$. You learned that the quotient of one number divided by another number is called a ratio (rational number). It may be written as the fraction $\frac{a}{b}$ (where $b \neq 0$).

Definition. The Ratio of a number a to a number b $(b \neq 0)$ is the quotient $\frac{a}{b}$. (Sometimes this is written $a : b$.)

The definitions and the examples show that a ratio compares just two numbers. In the examples there were two sets of numbers. One set was made up from the lengths of the shadows, and the other set was made up from the heights. Notice that when the fraction name of a ratio is written, care must be taken to write the proper numerals in the proper places. The ratio of a to b is $\frac{a}{b}$ where the first number is the numerator and the second number is the denominator.

The ratio of 32 to 48 is $\frac{32}{48}$. The simplified form of $\frac{32}{48}$ is $\frac{2}{3}$, so the ratio of 32 to 48 is $\frac{2}{3}$.

Exercises 10-1

1. Find the ratio of Helen's height to Laurie's height. Helen is 48 inches tall, and Laurie is 64 inches tall.

   $\frac{48}{64} = \frac{3}{4}$

2. Josephine has 90 cents and Martin has 30 cents. What is the ratio of Josephine's amount of money to Martin's?

   $\frac{90}{30} = \frac{3}{1}$

3. Marc walked 5 miles and Jerry walked 3 miles. What is the ratio of the distance Jerry walked to the distance Marc walked?

   $\frac{3}{5}$

4. What is the ratio of Marc's distance to Jerry's distance in Exercise 3?

   $\frac{5}{3}$

5. Bus fare ten years ago was 10 cents. Now it is 30 cents. Find the ratio of the present fare to the former fare.

   $\frac{30}{10}$ or $\frac{3}{1}$

6. How much did the fare in Exercise 5 increase? Find the ratio of the increase to the former fare.

   $\frac{20}{10}$ or $\frac{2}{1}$

7. During a sale, \$8.00 shoes were sold for \$6.00. Find the ratio of the decrease in the cost to the original price.

   $\frac{2}{8}$ or $\frac{1}{4}$

8. Write the ratio of (a) 2 to 9 (b) 48 to 40
(c) 175 to 125 (d) 65 to 100.

9. Write an expression for the ratio of c to d. $\frac{c}{d}$

10. Write an expression for the ratio of s to 66. $\frac{s}{66}$

11. Write an expression for the ratio of 2k to 7m. $\frac{2k}{7m}$

12. Write an expression for the ratio of 40 to w. $\frac{40}{w}$

10-2. Proportion.

Suppose that Tony's older brother was 66 inches tall. Is it possible to find the length of his shadow if Tony had measured it when he made the other measurements?

To answer this question we shall write a fraction for the older brother's ratio and state that it is the same as the other ratios. We may use s for the number of inches in his shadow length which we wish to find

$$\frac{s}{66} = \frac{2}{3}$$

Now we have two fractions for the same number. We know how to express a rational number in different ways. It will help us to express $\frac{2}{3}$ as a fraction with denominator 66.

$$\frac{2}{3} = \frac{2 \cdot 22}{3 \cdot 22} = \frac{44}{66}$$

We can use $\frac{44}{66}$ instead of $\frac{2}{3}$ and write

$$\frac{s}{66} = \frac{44}{66}$$ (The denominators are the same.)

This tells us that $s = 44$ (The denominators, then, are the same.)

The older brother's shadow would have been 44 inches long.

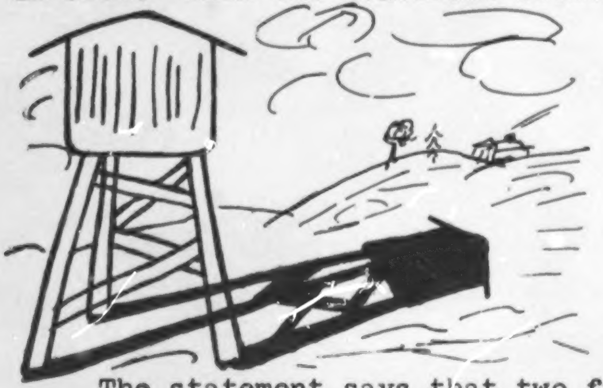
The statement $\frac{s}{66} = \frac{2}{3}$ above states that two ratios are equal. A statement that two ratios are equal is called a proportion. In a situation like this we say that the physical quantities measured by the numbers are proportional to one another. For example, the cost of gasoline is proportional to the number of gallons bought. The number of miles a car travels at a certain speed is proportional to the number of hours it takes. The amount of luxury tax charged is proportional to the price of the article. Name some physical quantities you think of which are proportional to one another.

In each case above, the shadow length is proportional to the height. For each pair of measures in Tony's table, the shadow length is proportional to the height and the ratio is $\frac{2}{3}$.

Later in the day, the ratio of shadow length to height changed to $\frac{5}{7}$. At this time the shadow of a water tower measured 21 feet. Then the statement about the height, w, of the water tower

$$\frac{21}{w} = \frac{5}{7}$$

is a proportion. We wish to find the height. To solve problems like this we need a method for finding a number which we can substitute for w to make the statement true.



The statement says that two fractions name the same rational number. From Chapter 6 we know the Comparison Rule:

If $\frac{a}{b} = \frac{c}{d}$ and $b \neq 0$, $d \neq 0$, then $a \cdot d = b \cdot c$.

We use this rule to solve our problem

$$\frac{21}{w} = \frac{5}{7}$$

$21 \cdot 7 = 5 \cdot w$ Comparison Rule

$147 = 5 \cdot w$

$\frac{147}{5} = w$ Definition of a rational number

$29\frac{2}{5} = w$ The water tower is about 29 feet in height.

The problem of the older brother's shadow can be solved in the same way.

$$\frac{s}{66} = \frac{2}{3}$$

$3 \cdot s = 2 \cdot 66$ State the property used here

$3 \cdot s = 132$ State the property used here

$s = 44$

The older brother had a 44 inch shadow.

Exercises 10-2a.

1. Find the ratio of the first number to the second.

		shadow	height	ratio
a.	Garage:	3 feet	10 feet	?
b.	Clothes pole:	36 inches	96 inches	?
c.	Fence:	11 inches	30 inches	?

2. In a class there are 36 children of whom 12 are girls.
    a. How many boys are in the class?
    b. What is the ratio of the number of girls to the total number of children in the class?
    c. What is the ratio of the number of boys to the total number of children in the class?
    d. What is the ratio of the number of girls to the number of boys?
3. In another class, the ratio of the number of girls to the number of boys is the same as in the class of Exercise 2. This class has 36 girls. How many boys does it have?
4. What number substituted for n will make the statement true?
    a. $\frac{n}{18} = \frac{4}{9}$ b. $\frac{56}{n} = \frac{7}{3}$ c. $\frac{16}{5} = \frac{96}{n}$

5. What number substituted for s will make the statement true?

a. $\frac{20}{8} = \frac{s}{6}$ b. $\frac{30}{14} = \frac{90}{s}$ c. $\frac{s}{42} = \frac{36}{27}$

6. Joyce has a picture 4 inches wide and 5 inches long. She wants an enlargement that will be 10 inches wide. How long will the enlarged print be?

7. Mr. Stephens was paid $135 for a job he finished in 40 hours. At this rate how much should he be paid for 60 hours of work?

8. A recipe for 30 cookies listed the following amounts

1 cup butter
$\frac{2}{3}$ cup sugar
2 eggs
$1\frac{1}{2}$ cups flour
1 tsp. vanilla

(a) Re-write the recipe to make 90 cookies.

(b) Suppose you wish to make 45 cookies using the original recipe. What ratio should you use? What amount of each item should you take?

9. What is the cost of 10 doughnuts at 50 cents a dozen? Did you think of it this way, where c is the number of cents 10 doughnuts cost? $\frac{c}{50} = \frac{10}{12}$

10. What is the cost of 12 candy bars at 3 for 25¢ ?

11. What is the cost of 2500 bricks at $14 per 1000 bricks?

12. The following table lists pairs of numbers v and w. In each pair the ratio of v to w is the number $\frac{6}{7}$. Copy and complete the table.

	v	w	ratio $\frac{v}{w}$ (simplest form)
a.	12	14	$\frac{6}{7}$
b.		21	
c.	30		
d.		100	
e.	100		

13. State which of the following ratios are equal. Use $a \cdot d = b \cdot c$ as your test.

a. $\frac{10}{5}$, $\frac{20}{10}$

b. $\frac{6}{3}$, $\frac{16}{9}$

c. $\frac{48}{16}$, $\frac{42}{14}$

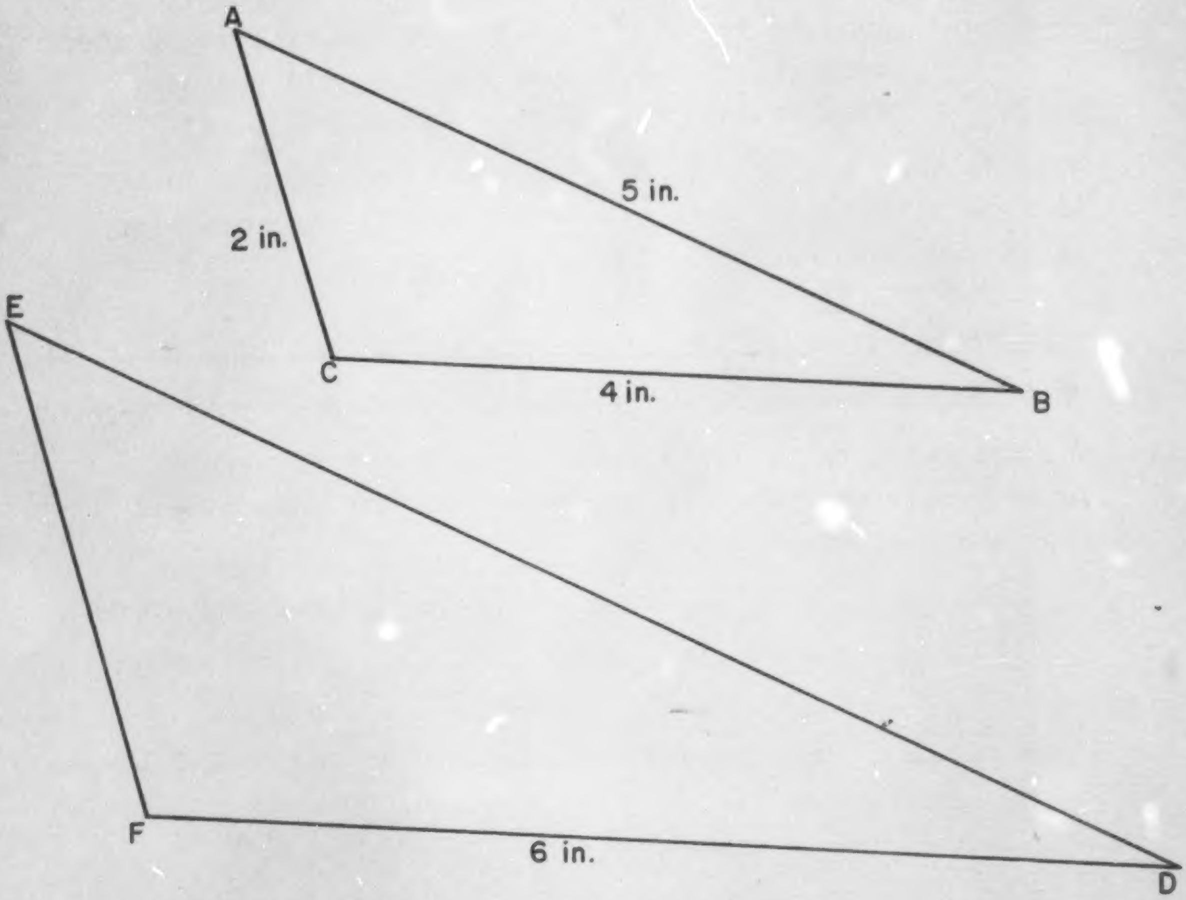
d. $\frac{68}{17}$, $\frac{76}{19}$

e. $\frac{15}{13}$, $\frac{45}{39}$

*14. When triangles are the same shape, but not the same size, they are called similar triangles. The measures of the lengths of corresponding sides of similar triangles form equal ratios. Triangles ABC and DEF are similar.

a. Find the length of side $\overline{EF}$.

b. Find the length of side $\overline{DE}$.



*Multiple ratios

Sometimes it is natural and desirable to compare more than just two quantities. For example, a mixture of nuts calls for 5 pounds of peanuts, 2 pounds of cashews, and 1 pound of pecans. Here the ratio of the number of pounds of peanuts to the number of pounds of cashews is $\frac{5}{2}$ or 5 to 2, and the ratio of the number of pounds of cashews to the number of pounds of pecans is $\frac{2}{1}$ or 2 to 1. This may be stated briefly as 5 to 2 to 1. Suppose a grocer wants to make up 24 pounds of this mixture. How many pounds of peanuts, cashews, and pecans should he use? To determine your answer, answer these questions first:

(1) If 5 pounds of peanuts, 2 pounds of cashews, and 1 pound of pecans are mixed together, how many pounds of nuts are there?

(2) What is the ratio of the number of pounds of peanuts to the total number of pounds?

(3) Since this ratio will be the same in the mixture whose total weight is 24 pounds, how many pounds of peanuts are required?

(4) Answer questions (2) and (3) when "peanuts" are replaced by "cashews."

(5) Answer questions (2) and (3) when "peanuts" are replaced by "pecans."

*Exercises 10-2b

1. A nut mixture of peanuts, cashews, and pecans in the ratio 5 to 3 to 2 is to total 100 pounds. How many pounds of each kind of nuts will the grocer need?
2. A nut mixture, with the same ingredients in the same ratio as above, is to total 56 pounds. How many pounds of each kind of nuts will the grocer need?

3. If a nut mixture with the same ingredients, in ratio 3: 2: 1, has 30 pounds of peanuts, how many pounds of cashews and pecans will be needed? What will the total weight be?

10-3. Percent

Many of you are familiar with the word "percent," and you may know something about its meaning. When your teacher says "90 percent of the answers on this paper are correct," do you know what he means? The word "percent" comes from the Latin phrase "per centum," which means "by the hundred." If the paper with 90 of the answers correct has 100 answers then 90 answers out of the 100 are correct. The ratio $\frac{90}{100}$ and be used instead of the phrase "90 percent" to describe the fraction of the answers which are correct. The word percent means hundredths. The denominator in the fraction $\frac{90}{100}$ may be written out in numerals, or indicated by the decimal point.

$$\begin{aligned} 90 \text{ percent} &= \frac{90}{100} = .90 \\ &= 90 \times \frac{1}{100} = 90 \times 0.01 \end{aligned}$$

For convenience a symbol, %, is used for the word "percent." This symbol is a short way of saying "times $\frac{1}{100}$" or "times 0.01."

$$90\% = 90 \times \frac{1}{100} \quad \text{or} \quad 90 \times 0.01$$

$$16\% = 16 \times \frac{1}{100} \quad \text{or} \quad 16 \times 0.01$$

You have learned that there are many ways to write the name of a number. For example:

$$1 = \frac{5}{5} = \frac{10}{10} = \frac{100}{100} = 1.0 = 1.00$$

$$2 = \frac{6}{3} = \frac{50}{25} = \frac{62}{31} = \frac{200}{100} = 2.00$$

$$\frac{1}{2} = \frac{2}{4} = \frac{9}{18} = \frac{50}{100} = .50$$

The percent symbol provides another way to write the name of a number.

$$1 = \frac{100}{100} = 100 \times \frac{1}{100} = 100\%$$

$$2 = 2.00 = 200 \times 0.01 = 200\%$$

$$\frac{1}{2} = \frac{50}{100} = 50 \times \frac{1}{100} = 50\%$$

Exercises 10-3a

1. Using squared paper draw a large square so that the interior is divided into 100 small squares. Write the letter A in each of 10 small squares. Write the letter B in each of 20% of the squares. Write the letter C in each of 35% of the squares. Write the letter D in each of 30 of the squares. Write the letter X in the remainder of the squares.
   a. In what fractional part of the squares is the letter A?
   b. In what percent of the squares is the letter A?
   c. In how many squares is the letter B?
   d. In what fractional part of the squares is the letter B? Give your answer in the simplest form.
   e. In how many squares is the letter C?
   f. In what fractional part of the squares is the letter C? Give your answer in the simplest form.
   g. In what fractional part of the squares is the letter D?
   h. In what percent of the squares is the letter D?
   i. In what fractional part of the squares is the letter X?
   j. In what percent of the squares is the letter X?
2. What is the sum of the fractions in parts a, d, f, g, i, of problem 1?
3. What is the sum of the percents of the squares containing the letters A, B, C, D, X?

4. Copy the table below and fill in the blank spaces

a.	$\frac{1}{4}$	$\frac{25}{100}$	$25 \times \frac{1}{100}$	25%
b.	$\frac{2}{5}$	$\frac{40}{100}$	$40 \times \frac{1}{100}$	____
c.	$\frac{3}{4}$	$\frac{75}{100}$	____	____
d.	$\frac{7}{10}$	____	____	____
e.	$\frac{8}{5}$	____	____	____
f.	4	$\frac{400}{100}$	____	____
g.	0.3	0.30	30×0.01	____
h.	0.9	____	____	____
i.	0.78	0.78	____	____

5. Jean has a weekly allowance of 50 cents. One week she spends 12 cents for a pencil, 10 cents for an ice cream cone, 15 cents for Sunday School collection, and puts the rest in her piggy bank.

   a. Express each amount as a ratio of the whole, and also as a percent.

   b. Find the sum of the ratios.

   c. Find the sum of the percents.

   d. What check on the separate answers above do you observe?

6. The following boys and girls live on the same street.

Robert Joan Mary Dwight Tony
Sarah Raymond Gloria Lars Max

   a. What percent are boys?

   b. What percent are girls?

   c. What is the sum of the two percents?

7. What do we mean when we say each of the following?

   a. This fabric is 100% wool.

   b. We have had 100% attendance for a week.

   c. 100% of the members were in favor of a picnic.

Percent is a convenient tool for giving information involving ratios. Athletic standings often are given in percent. Two seventh grade pupils discovered the reason for this use of percent as they talked about their neighborhood baseball teams. The boys were discussing the scores of their baseball teams. In the Little League the first boy's team won 15 games out of 20 games played. The other boy's team won 18 out of 25 games. Which team had the better record? The second team won 3 more games, but the first team played fewer games. Look at the ratios of the number of games won to the number of games played for each team. The ratios $\frac{15}{20}$ and $\frac{18}{25}$ are not easy to compare at a glance. Percent makes the comparison easy.

The first team won $\frac{15}{20}$ of the games played.

$\frac{15}{20} = \frac{75}{100} = 75\%$. The first team won 75% of the games played. The second team won $\frac{18}{25}$ of the games played. $\frac{18}{25} = \frac{72}{100} = 72\%$. The second team won 72% of the games played.

The first team which won 75% of its games had a higher standing than the second team which won 72% of its games. We could say that $72\% < 75\%$, or $75\% > 72\%$.

Information from business, industry, school, Scouts and similar sources is often given in percent. It is often more convenient to refer to the information at some later time if it is given in percent than if it is given in another form.

A few years ago the director of a Boy Scout camp kept some records for future use. Some information was given in percent, and some was not. The records gave the following items of information.

(1) There were 200 boys in camp.

(2) One hundred percent of the boys were hungry for the first dinner in camp.

(3) Fifteen percent of the boys forgot to pack a toothbrush, and needed to buy one at camp.

(4) On the second day in camp 44 boys caught fish.

(5) One boy wanted to go home the first night.

(6) A neighboring camp director said "Forty percent of the boys in camp will learn to swim this summer. We shall teach 32 boys to swim."

From (1) and (2), how many hungry boys came to dinner the first day?

Of course we know without computation that 100% of anything is all of it. Then 100% of 200 is 200. The answer is that 200 hungry boys came to dinner the first day.

From (3), how many extra toothbrushes were needed?

$$15\% \text{ means } 15 \cdot \frac{1}{100} \text{ or } \frac{15}{100}$$

The ratio of boys without toothbrushes to the total number of boys in camp is $\frac{?}{200}$. This can be written $\frac{n}{200}$ where n is the number of toothbrushes needed.

The ratio $\frac{n}{200}$ equals the ratio $\frac{15}{100}$ since 15% of the boys forgot to pack a toothbrush.

$$\frac{n}{200} = \frac{15}{100}$$

$$100\,n = 15 \cdot 200 \qquad \textbf{Comparison Rule}$$

$$n = \frac{15 \cdot 200}{100} \qquad \textbf{Definition of a rational number.}$$

$$n = 30$$

30 toothbrushes were needed.

Statement (4) says that 44 boys caught fish. The percent of boys who caught fish can be found. The ratio of the 44 boys to the 200 boys in camp is $\frac{44}{200}$. If y percent caught fish, this ratio is $y \cdot \frac{1}{100}$ or $\frac{y}{100}$.

$$\frac{44}{100} = \frac{y}{100}$$

$$44 \cdot 100 = 200\,y$$

$$\frac{44 \cdot 100}{200} = y$$

$$\frac{44 \cdot 100}{2 \cdot 100} = y$$

$$22 = y$$

Then y percent is 22 percent or 22%.

22% of the boys caught fish on the second day.

From statement (5) we can find the percent of the boys who were homesick. One boy wanted to go home the first night.
If v percent wanted to go home, we have

$$\frac{1}{200} = \frac{v}{100}$$

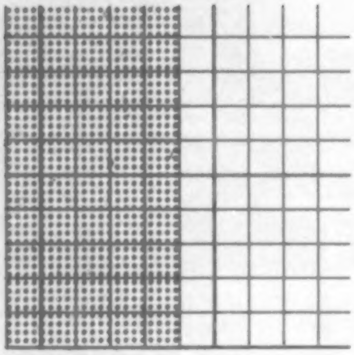
$$100 = 200 \cdot v$$

$$\frac{100}{200} = v$$

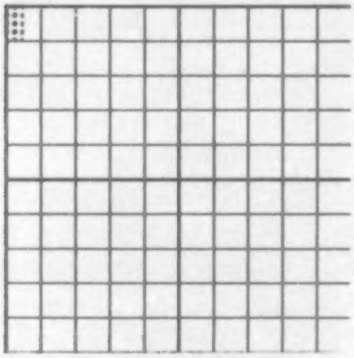
$$\frac{1}{2} = v$$

Then $\frac{1}{2}$% of the boys wanted to go home. This may be read, "one-half percent" or "one-half of one percent".

Be sure that you know the difference between the rate $\frac{1}{2}$ and the rate $\frac{1}{2}\%$.



$\frac{1}{2}$



$\frac{1}{2}\%$

Note that $\frac{1}{2}\%$ may also be written 0.5% which is read, "five-tenths percent" or "five-tenths of one percent."

Statement (6) of the camp problem says that 40% of the boys in a neighboring camp were 32 boys. In symbols we write:

$$\frac{40}{100} = \frac{32}{n}$$

where n is the total number of boys in the neighboring camp.

$$40n = 3200$$

$$n = \frac{3200}{40}$$

$$n = 80$$

There were 80 boys in the neighboring camp.

Now you have had examples of all the kinds of problems which will probably arise for you in percent.

Exercises 10-3b

1. Usually 8% of the pupils who buy season passes to school events fail to use them. If 250 pupils bought passes, how many pupils probably will not use them?
   a. Explain where $\frac{8}{100}$ comes from in this problem.
   b. Explain where $\frac{n}{250}$ comes from, if n is the number of pupils who probably will not use their passes.
   c. Explain why $\frac{8}{100} = \frac{n}{250}$.
   d. Find the value of n which makes the statement in (a) true.
   e. Would you accept an answer of 150 as sensible? Why?
2. There were 300 boys in the school and 120 signed up for track practice. What percent signed up for track?
   a. What ratio of the boys signed up for track?
   b. Write an expression for x percent, using the symbol $\%$.
   c. Write a fraction for (b) using 100 as the denominator.
   d. Write the statement that the expressions in (a) and (c) mean the same number.
   e. Find the value of x which makes statement in (d) true.
   f. Answer the question in the problem.
      Is your answer a sensible answer for the problem?
3. A Little League team won 3 out of the first 5 games played.
   a. What percent of the first 5 games did the team win?
   b. What percent of the first 5 games did the team lose?
   c. What percent of the games played did the team win or lose?
4. Later in the season the team in Exercise 3 won 8 out of 16 games played.
   a. What was the percent of games won at this time?
   b. Did the percent of games won increase or decrease?

5. At the end of the season, the team in Exercise 3 had won 26 games out of 40.
   a. What percent of the games played did the team win by the end of the season?
   b. How does this percent compare with the other two?

Exercises 6-12 all refer to the same junior high school.

6. There are 600 seventh grade pupils in a junior high school. The principal plans to divide the pupils into 20 sections of equal size.
   a. How many pupils will be in each section?
   b. What percent of the pupils will be in each section?
   c. How many pupils are 1% of the number of pupils in the seventh grade?
   d. How many pupils are 10% of the number of pupils in the seventh grade?

   Answers to (c) and (d) may help you with other answers.

7. Suppose one section contains 36 pupils. What percent of the seventh grade pupils are in that section?

8. One hundred fifty seventh grade pupils come to school on the school bus.
   a. What percent of the seventh grade pupils come by school bus?
   b. What percent of the seventh grade pupils come to school by some other means?

9. In a class of 30 pupils, 3 were tardy one day.
   a. What fractional part of the class do the tardy pupils represent?
   b. What percent of the class was tardy?

10. One day the Principal said, "Four percent of the ninth graders are absent today." The list of absentees had 22 names of ninth graders on it. From these two pieces of information find the number of ninth grade pupils in the class.

10-4. Decimals and Percent

Sometimes it is convenient to write a percent as a fraction in the form $\frac{a}{100}$ and sometimes the decimal form is convenient. It is necessary to be skillful in writing all possible forms in order to be free to choose the easy one.

To change a percent to a decimal or a fraction in simplest form, you may follow these steps:

Example 1. $65\% = 65 \times 0.01 = 0.65$

or $65\% = 65 \times \frac{1}{100} = \frac{65}{100} = \frac{13}{20}$

Example 2. $12\frac{1}{2}\% = 12.5 \times 0.01 = 0.125$

or $12\frac{1}{2}\% = 12\frac{1}{2} \times \frac{1}{100} = \frac{25}{2} \times \frac{1}{100} = \frac{25}{200} = \frac{1}{8}$

To change a decimal fraction to a percent you may follow these steps:

Example 1. $0.23 = 23 \times 0.01 = 23\%$

or $0.23 = \frac{23}{100} = 23 \times \frac{1}{100} = 23\%$

Example 2. $0.05 = 5 \times 0.01 = 5\%$

or $0.05 = \frac{5}{100} = 5 \times \frac{1}{100} = 5\%$

Example 3. $0.472 = 47.2 \times 0.01 = 47.2\%$

or $0.472 = \frac{472}{1000} = \frac{59}{125}$

To change a fraction of the form $\frac{a}{b}$ to a percent, you may follow these steps:

Example 1. $\frac{1}{3} = \frac{1}{3} \times \frac{100}{100} = \frac{100}{3} \times \frac{1}{100} = 33\frac{1}{3} \times \frac{1}{100} = 33\frac{1}{3}\%$

or $\frac{1}{3} = 0.33 \ldots = 33\frac{1}{3}\%$

Example 2. $\frac{5}{13} = \frac{5}{13} \times \frac{100}{100} = \frac{500}{13} \times \frac{1}{100} = 38\frac{6}{13} \times \frac{1}{100} = 38\frac{6}{13}\%$

or $\frac{5}{13} \approx .384 \approx 38 \times 0.01 = 38\%$

Note that it is necessary to use the "approximately equal" sign. Since the decimal fraction does not terminate, $\frac{5}{13}$ is not exactly the same as 0.384. The equal sign cannot be used. The two wavy lines ≈ are used to mean "approximately equal."

Exercises 10-4a

1. Write the following fractions in the form indicated:

   a. $\frac{3}{4} = \frac{\quad}{100} = \_\_\_\%$

   b. $\frac{4}{5} = \frac{\quad}{100} = \_\_\_\%$

   c. $\frac{7}{10} = \frac{\quad}{100} = \_\_\_\%$

   d. $\frac{19}{25} = \frac{\quad}{100} = \_\_\_\%$

   e. $\frac{11}{20} = \frac{\quad}{100} = \_\_\_\%$

   f. $\frac{9}{25} = \frac{\quad}{100} = \_\_\_\%$

   g. $\frac{17}{20} = \frac{\quad}{100} = \_\_\_\%$

   h. $\frac{3}{5} = \frac{\quad}{100} = \_\_\_\%$

2. Write the following fractions in decimal form. Round to the nearest hundredth.

   a. $\frac{1}{3}$

   b. $\frac{5}{6}$

   c. $\frac{3}{8}$

   d. $\frac{7}{9}$

   e. $\frac{6}{11}$

   f. $\frac{43}{49}$

   g. $\frac{86}{127}$

   h. $\frac{265}{591}$

3. Write the percent form of the decimal fraction in Exercise 2. Notice that when you round to the nearest hundredth in a decimal fraction, the percent form is rounded to the nearest whole percent.

4. Write the following fractions in decimal form. Round to the nearest thousandth.

   a. $\frac{3}{8}$

   b. $\frac{2}{3}$

   c. $\frac{8}{9}$

   d. $\frac{29}{30}$

   e. $\frac{5}{6}$

   f. $\frac{58}{65}$

   g. $\frac{126}{490}$

   h. $\frac{387}{462}$

5. Write the percent form of the decimal fraction in Exercise 4. To what place do you round a decimal fraction when you wish the percent form to be rounded to the nearest tenth of a percent?

There are four commonly used ways of writing the same ratio. You should be able to use any of these. They are: the fraction in simplest form, the fraction with 100 as the denominator, the decimal fraction, and the percent.

If one of these names for a number is shown, you can find the other three.

Example 1. $0.28 = \frac{28}{100} = 28 \quad = \frac{7}{25}$

$0.15 = \ ? \ = \ ? \ = \ ?$

Example 2. $72 \quad = 0.72 = \frac{72}{100} = \frac{18}{25}$

$48 \ = \ ? \ = \ ? \ = \ ?$

Example 3. $\frac{2}{5} = \frac{40}{100} = 0.40 = 40$

$\frac{14}{25} = \ ? \ = \ ? \ = \ ?$

Example 4. $\frac{5}{6} = \frac{83\frac{1}{3}}{100} = 83\frac{1}{3}\% = 0.83\ldots$

$\frac{2}{3} = \ ? \ = \ ? \ = \ ?$

Example 5. $\frac{3}{100} = \frac{3}{100} = 3\% = .03$

$\frac{19}{100} = ? = ? = ?$

Notice that in Example 5 there were only 3 different number names since the form $\frac{3}{100}$ has the denominator 100.

Exercises 10-4b

1. Fill in the missing names of numbers in the chart below. The completed chart will be helpful to you in future lessons.

Fraction Simplest form	Hundred as denominator	Decimal	Percent
a. $\frac{1}{2}$	$\frac{50}{100}$	.50	50%
b. $\frac{1}{4}$			
c.	$\frac{75}{100}$		
d.		.20	
e.			40%
f.	$\frac{60}{100}$		
g. $\frac{4}{5}$			
h.		.33...	
i.	$\frac{70}{100}$		
j.		.66...	
k. $\frac{3}{10}$			
l.		.10	

Fraction Simplest form	Hundred as denominator	Decimal	Percent
m.			90%
n. $\frac{1}{8}$			
o.	$\frac{300}{100}$		
p.		.375	
q.			150%
r.	$\frac{62.5}{100}$		
s.		.01	
t. $\frac{7}{8}$			
u.			100%
v.	$\frac{16\frac{2}{3}}{100}$		
w. $\frac{5}{6}$			

2. Draw a number line and at points about 5 inches apart mark 0 and 100%.

   a. Locate the following percents from Exercise 1 on this line. Estimate their positions and mark the percent names below the line. a, b, c, d, e, f, g, h, n, p, t, w.

   b. Mark these fractions above the appropriate points on the line.

   $\frac{1}{6}, \frac{1}{5}, \frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}, \frac{3}{4}, \frac{3}{5}, \frac{7}{8}, \frac{4}{5}, \frac{2}{3}, \frac{3}{8}.$

3. Using squared paper, draw 5 large squares each containing 100 small squares. By proper shading show the percents in b, d, ℓ, p, s of problem 1.

4. What fraction in simplest form is another name for

a. 32% b. 90% c. 120%

5. Give the percent names for the following numbers.

a. $\frac{13}{25}$ b. $\frac{7}{20}$ c. $\frac{19}{20}$ d. $\frac{3}{10}$

10-5. Applications of Percent

Percent is used to express ratios of numerical quantities in everyday experience. It is important for you to understand the idea of percent, the way it is written, and, also, to be accurate in working with percent.

Budgets

A budget is a plan for getting the most value for money earned and then spent. Families use budgets to help them work out how their money will be spent for food, housing, personal needs, savings and other things. Governments, businesses, and school systems all have budgets to show how they plan to spend their incomes. Boys and girls in school have budgets, too, if they want to get the most satisfaction from the money they spend.

Suppose a family had a monthly income of $410 after taxes. The family budget allowed 32% for food. How much money is 32% of $410?

The problem is to find m, the number of dollars allowed for food. The ratio $\frac{32}{100}$ is known. Tell why this is known.

The ratio $\frac{m}{410}$ is another way to express the ratio of the number of dollars allowed for food to the number of dollars of income. Then

1. $\frac{m}{410} = \frac{32}{100}$ Tell why

$100m = 32(410)$ Explain this step

2. $m = \frac{14120}{100}$ Explain this step

$m = 141.20$

$141.20 is the amount allowed for food.

If the ratio 32% is written in decimal form as 0.32, the problem looks like this:

1. $\frac{m}{410} = 0.32$

$m = 0.32(410)$ Property of rational numbers.

$m = 141.20$

$141.20 is the amount allowed for food.

Perhaps you noticed that in step 2 the expression on the right is the product of the ratio (0.32 or $\frac{32}{100}$) and the total amount 410. It is possible to think of 32% of 410 as 0.32×410 since in a "fraction of a number" the "of" means times. The ratio method, however, is useful in every type of percent problem. It is preferred for this reason.

Suppose the same family rents an apartment for $78 per month. What percent of the family income of $410 per month is spent for rent? The problem is to find the percent of income spent for rent. Let us call this ratio $\frac{r}{100}$. The known ratio is $\frac{78}{410}$. Notice that this is about $\frac{80}{400}$ or about $\frac{1}{5}$. A good estimate then, is about 20%. It is always sensible to estimate the answer before you work the problem.

$$\frac{r}{100} = \frac{78}{410}$$

$$r = \frac{78 \times 100}{410}$$

$$r \approx 19.2$$

The percent of income spent for rent is about 19%.

The wavy lines ≈ mean that r is not exactly the same as 19.2 since the decimal does not terminate in the tenths place. The value 19.2 is an approximation for r. The rounded answer 19% is close enough for practical purposes.

Exercises 10-5a

Suggested Budget for a Family of Four

Take-home pay	\$300 per mo.	\$350 per mo.	\$450 per mo.	\$550 per mo.
Food	30%	27%	22%	22%
Housing	30%	28%	28%	28%
Personal needs	13%	13%	12%	12%
Transportation	7%	7%	9%	9%
Savings, Taxes	5%	10%	16%	16%
Others	15%	15%	13%	13%
Total	100%	100%	100%	100%

1. Why should each column total 100%.
2. What percent of the \$450 budget is allowed for food?
3. a. Write the ratio in Exercise 2 as a fraction with 100 as the denominator.
   b. Use f to stand for the number of dollars allowed for food, and write the ratio of f to \$450.
   c. 22% is about $\frac{1}{5}$. Use this to estimate amount of money spent on food from a \$450 income.
   d. Use the ratio in a and b to write a proportion and find the number, f.
   e. Check your answer with the estimate in c.
4. a. Write the percent of the \$550 income which is allowed for the item called Savings, Taxes.
   b. Find the number of dollars of the \$550 income which are allowed for Savings, Taxes.
   c. Notice that 16% is between 10% and 20%. Then it is between $\frac{1}{10}$ and $\frac{1}{5}$. How can you use this information to check your answer?

5. What amount, according to the table, is allowed for Personal Needs from an income of $300?

6. Find the amount allowed for Transportation when the income is $350.

7. Find the amount allowed for Housing when the income is $450 a month.

8. Find the amount allowed for Savings, Taxes when the income is $300 per month.

9. The Hughes family of 4 has a monthly income of $575 after taxes. The amount this family spends for housing averages $150 a month. Find the percent of their income they use and compare it with the percent allowed in the table for an income of $550. Round the answer to the nearest whole percent.

*10. An apartment for a family of 5 rents for $95 a month. If the family plans to spend 25% of the monthly income for rent, what monthly income does the family need?

Commissions and discounts

People who work as salesmen often are paid a commission instead of a salary. The money they earn depends upon the sales they make. A book salesman, for example, may be paid a commission of 25% of the selling price of the books he sells. If he sells $12,000 worth of books to a school, his commission is 25% of $12,000.

If c represents the number of dollars earned, then the ratio of the number of dollars earned to the number of dollars of sales is $\frac{c}{12,000}$. The commission ratio is $\frac{25}{100}$.

$$\frac{c}{12,000} = \frac{25}{100}$$
$$100c = 25(12,000)$$
$$c = \frac{300,000}{100}$$
$$c = 3,000$$

The salesman earns $3,000 on his sales of $12,000.

The numbers in this problem are easy and you probably could think 25% is $\frac{1}{4}$ and $\frac{1}{4}$ of $12,000 is $3,000. Other situations will use numbers which do not lend themselves to short cuts. Hence, the above example was solved in the general way.

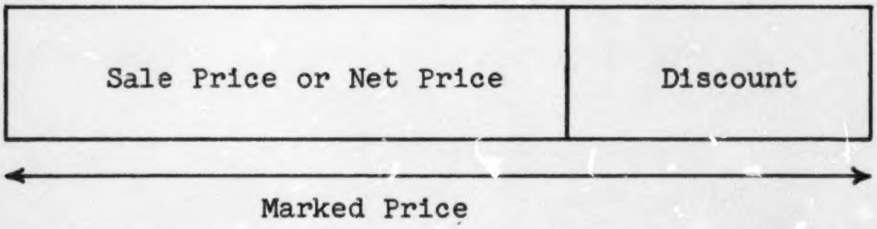
Sometimes the percent of the selling price which gives the commission is called the rate of the commission.

Definition: Commission is the payment, often based on a percent of the selling price, that is paid to a salesman for his services.

Merchants sometimes sell articles at a discount. During a sale, an advertisement stated "All coats will be sold at a discount of 30%." A coat marked $70.00 then has a discount of 30% of $70.00 or $21.00. The sale price (sometimes called the net price) is $70.00 - $21.00 or $49.00.

Definition: Discount is the amount subtracted from the marked price.

Definition: Sale price or net price is the marked price less the discount.



Example: Find the sale price of a coat marked $70 with a discount of 30%. Choose d to represent the number of dollars in this discount.

$$\frac{d}{70} = \frac{30}{100}$$

$$100d = 2100$$

$$d = \frac{2100}{100}$$

$$d = 21$$

The discount is $21

The sale price is $70 - $21 or $49.

Exercises 10-5b

In the following problems it may be necessary to round some answers. Round money answers to the nearest cent, and round percent answers to the nearest whole percent.

1. On an examination there was a total of 40 problems. The teacher considered all of the problems of equal value, and assigned grades by percent. How many correct answers were indicated by the following grades?

   a. 100% b. 80% c. 50% d. 65%

2. What percent did the teacher in Exercise 1 assign for the following papers?

   a. All problems worked, but 10 answers wrong.

   b. 36 problems worked, all answers correct.

   c. 20 problems worked, 2 answers wrong.

   d. 1 problem not answered, and 1 answer wrong.

3. If the sales tax in a certain state is 4% of the purchase price, what tax is collected on the following purchases?

   a. A dress selling for $17.50

   b. A bicycle selling for $49.50.

4. A real estate agent receives a commission of 5% for any sale that he makes. What is his commission on the sale of a house for $27,500?

5. The real-estate agent in Exercise 4 wishes to earn an annual income of at least $9,000 from his commissions. To earn this income, what must his yearly sales total?

6. A salesman who sells vacuum cleaners earns a commission of $25.50 on each one sold. If the selling price of the cleaner is $85.00 what is the rate of commission for the salesman?

7. Sometimes the rate of commission is very small. Salesmen for heavy machinery often may receive a commission of 1%. If, in one year, such a salesman sells a machine to an industrial plant for $658,000 and another machine for $482,000, has he earned a good income for the year? What is the income?

8. A sports store advertised a sale of football equipment. The sale discount was 27%.
   a. What was the sale price of a football marked at $5.98?
   b. What was the sale price of a helmet if the marked price was $3.75?

9. In a Lincoln High School there are 380 seventh grade pupils, 385 eighth grade pupils, and 352 ninth grade pupils.
   a. What is the total enrollment of the school?
   b. What percent of the enrollment is in the seventh grade?
   c. What percent of the enrollment is in the eighth grade?
   d. What percent of the enrollment is in the ningh grade?
   e. What is the sum of the answers to b, c, and d?

10. Mr. Martin kept a record of the amounts of money his family paid in sales tax. At the end of one year he found that the total was $96.00 for the year. If the sales tax rate was 4%, what was the total amount of taxable purchases made by the Martin family during the year?

Percents used for comparison

In problem 1 of Exercises 10-4b, some of the percents were not whole percents. The fraction $\frac{1}{8}$ changed to percent is $12\frac{1}{2}\%$. Also, you have used $\frac{1}{2}\%$ and have learned that $\frac{1}{2}\%$ may be read $\frac{1}{2}$ of 1%. Decimal percents such as 0.7% are often used, too.

$$0.7\ (0.7 \text{ of } 1\ ) = 0.7 \times \frac{1}{100} = \frac{0.7}{100} = \frac{7}{1000} = 0.007$$

$$\text{or} \quad .7 \times 0.01 = 0.007$$

These are all names for the same number. If we wish to find 0.7% of $300, we wish to find n so that

$$\begin{array}{lcl} \frac{n}{300} = \frac{0.7}{100} & \text{or} & \frac{n}{300} = 0.007 \\ & & n = 300(0.007) \\ 100n = 210 & & n = 2.10 \\ n = 2.10 & & 0.7\% \text{ of } \$300 \text{ is } \$2.10 \end{array}$$

0.7% of $300 is $2.10
(0.7% is less than 1%. Since 1% of $300 is $3.00, the answer $2.10 is sensible.)

Suppose that we wish to find 2.3% of $500.

$2.3\% = \frac{2.3}{100} = \frac{23}{1000} = 0.023$. Find the number x such that

$$\begin{array}{lcl} \frac{x}{500} = \frac{2.3}{100} & \text{or} & \frac{x}{500} = 0.023 \\ 100x = 1150 & & x = 500 \cdot 0.023 \\ x = 11.50 & & x = 11.50 \\ 2.3\% \text{ of } \$500 \text{ is } \$11.50 & & 2.3\% \text{ of } \$500 \text{ is } \$11.50 \end{array}$$

Baseball batting averages are found for each player by dividing the number of hits the player made by his number of times at bat. The division is carried to the nearest thousandth. The batting average, then, can be considered a percent expressed to the nearest tenth of a percent. If a player has 23 hits out of 71 times at bat, his batting average is $\frac{23}{71}$ or .324. This is 32.4%. In newspaper reports the decimal point in .324 is often omitted in order to save space.

Sometimes answers are called for to the nearest tenth of a percent. A teacher may be asked what percent of the marks of his pupils are B. Suppose he issued 163 marks, 35 of them B. He wishes to find x such that $\frac{35}{163} = \frac{x}{100}$.

$$\frac{35}{163} = \frac{x}{100} \qquad \text{or} \qquad \frac{35}{163} = \frac{x}{100}$$

$$x = \frac{35}{163} \cdot 100 \qquad 163x = 3500$$

$$x = 21.47\ldots \qquad x = 21.47\ldots$$

In the chapter on decimal notation you learned how to round decimals. If the percent is called for to the nearest tenth of a percent, 21.47... is rounded to 21.5%.

Exercises 10-5c

1. Find the batting averages of these four players. Who has the best average? Give answers as 3 place decimals and as percents.

Player	Times at bat	Hits
George	65	19
Max	70	22
Bill	73	21
Tom	60	21

2. A nut grower has found that 4.6% of the nuts that he grows can be expected to be below standard. At this rate, how many pounds of nuts will be below standard in a bag weighing 80 pounds? Express your answer to the nearest .1 pound.

3. When Mr. Davis built his new home, he found that the cost of the land was 18.5% of the cost of the house. If the land cost $3,330, what was the cost of the house?

4. The Lewis family has an income of $5,200 a year. If they save $450 a year, what percent of the income is this? Express your answer to the nearest .1%.

Percents of increase and decrease

Percent is used to indicate increase or a decrease in some quantity. Suppose that Central City had a population of 32,000 (rounded to the nearest thousand) in 1950. If the population increased to 40,000 by 1960, what was the percent of increase?

The "percent of increase" asks for the ratio of the increase to the original amount. In this case the increase is 40,000 - 32,000 or 8,000. The original population, or the population in the beginning, was 32,000. The problem is to find g so that

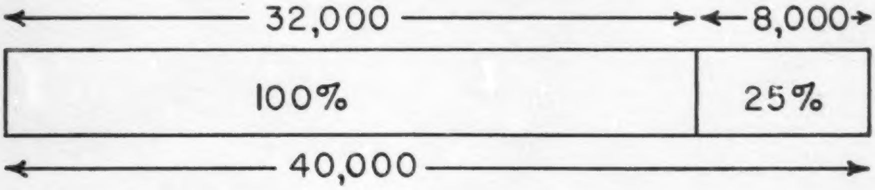
$$\frac{g}{100} = \frac{8,000}{32,000}$$

$$g = \frac{8,000 \times 100}{32,000}$$

$$g = 25$$

The percent of increase was 25%.

Notice that the percent of increase compares the actual increase to the earlier population number.



The 40,000 is made up of the 32,000 (100%) plus the increase of 8,000 (25%). The population of 40,000 in 1960 is 125% of the population of 32,000 in 1950.

Suppose that Hill City had a population of 15,000 in 1950. If the population in 1960 was 12,000, what is the percent of decrease? If x represents the percent of decrease, then

$$\begin{array}{r}15{,}000\\ -12{,}000\\ \hline 3{,}000\end{array}\quad \text{actual decrease}$$

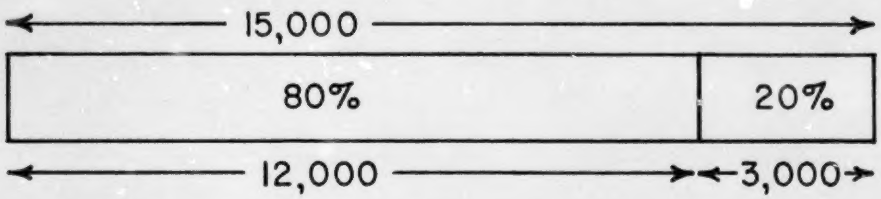
$$\frac{3{,}000}{15{,}000} = \frac{x}{100}$$

$$x = 100 \cdot \frac{3{,}000}{15{,}000}$$

$$x = 20$$

The percent of decrease was 20%

Notice that the population decrease is computed by comparing the actual decrease to the earlier population figure.



The 12,000 is the difference between 15,000 (100%) and the decrease of 3,000 (20%). The population of 12,000 in 1960 is 80% of the population of 15,000 in 1950.

If the rents in an apartment house are increased 5%, each tenant can compute his new rent. Suppose that a tenant is paying $80 for rent. What must he pay in rent after the increase? If x represents the increase in rent, then

$$\frac{x}{80} = \frac{5}{100}$$

$$x = 80 \cdot \frac{5}{100}$$

$$x = 4$$

The increase is $4.00

The new rent will be $80 + $4 = $84.

Exercises 10-5d

1. There were 176 pupils in 5 mathematics classes. The semester marks of the pupils were A 20; B 37; C 65; D 40; E 14. Give answers to the nearest tenth percent.
   - a. What percent of the marks were A?
   - b. What percent of the marks were B?
   - c. What percent of the marks were C?
   - d. What percent of the marks were D?
   - e. What percent of the marks were E?
   - f. What is the sum of the answers in parts a, b, c, d, e? Does this sum help you check your answers?
2. Bob's weight increased during the school year from 65 pounds to 78 pounds. What was the percent of increase?
3. During the same year, Bob's mother reduced her weight from 160 pounds to 144 pounds. What was the percent of decrease?
4. The enrollment in a junior high school was 1240 in 1954. In 1959 the enrollment had increased 25%. What was the enrollment in 1959?
5. Jean earned $14.00 during August. In September she earned only $9.50. What was the percent of decrease in her earnings?
6. A salesman of heavy machinery earned a commission of $4,850 on the sale of a machine for $970,000.
   - a. Find his rate of commission.
   - b. What can he expect as his commission for the sale of another machine for $847,500?
7. James was 5 ft. tall in September. In the following June his height was 5 ft. 5 in. Both heights were measured to the nearest inch. What was the percent of increase in his height?

8. Do you know what your height was at the beginning of the school year? Do you know what it is now? Do you know your weight at the beginning of the school year? Now?

   a. What is the percent of increase in your height since last September?

   b. What is the percent of increase in your weight since last September?

9. A baseball player named Walsh made 25 hits out of 83 times at bat. Another player named Schaefer made 42 hits out of 143 times at bat.

   a. What is the batting average of each player?

   b. Which player has the better record?

10. An elementary school had an enrollment of 790 pupils in September, 1955. In September, 1959, the enrollment was 1012. What was the percent of increase in enrollment?

11. On the first day of school a junior high school had an enrollment of 1050 pupils. One month later the enrollment was 1200. What was the percent of increase to the nearest tenth of a percent?

12. One week the school lunchroom received $450. The following week the amount was $425. What was the percent of decrease to the nearest tenth of a percent?

13. A baby's weight usually increases 100% in his first six months of life.

   a. What should a baby weigh at six months, if its weight at birth was 7 lb. 9 oz.?

   b. Suppose that the baby in a weighs 17 lb. at the age of six months. What is the percent of increase to the nearest percent?

14. During 1958 a family spent $1,490 on food. In 1959 the same family spent $1,950 on food. What was the percent of increase in the money spent for food to the nearest percent?

*15. During 1958 the owner of a business found that sales were below normal. The owner announced to his employees that all wages for 1959 would be cut 20%. By the end of 1959 the owner noted that sales had returned to the 1957 levels. The owner then announced to the employees that the 1960 wages would be increased 20% over those of 1959.

a. Which of the following statements is true?

(1) The 1960 wages are the same as the 1958 wages.

(2) The 1960 wages are less than the 1958 wages.

(3) The 1960 wages are more than the 1958 wages.

Interest

The charging and payment of interest is important in business affairs. You may have noticed this. Savings banks advertise that they pay 3% or 4% on savings. Lending companies advertise that they will lend money at 6% or 7%. The numbers change but the ideas of percent remain the same.

Everyone knows that if you live in a house or apartment which belongs to someone else you pay for this privilege. The fee you pay is called rent.

In the same way, if you arrange to use money which belongs to someone else, you pay for this privilege. The fee you pay is called interest. If you have money in a savings account you receive the interest, for then you are the lender. The period of time for which money is borrowed or loaned is usually considered to be based on a year. When a savings account pays 3%, it pays 3 %if the amount of money in the account each year.

The amount of money upon which the interest is paid is called the principal. The ratio of the interest to the principal is call called the rate of interest.

Example: At the end of the year Marie received $13.50 interest on her savings account of $450. Her bank paid 3% on savings for one year. $450 is the principal. $13.50 is the interest. 3% is the rate.

a. How do you find the rate when you know the interest and the principal? The rate is a percent. Let $\frac{r}{100}$ represent the ratio. Then

$$\frac{r}{100} = \frac{13.50}{450}$$

$$r = \frac{13.50 \times 100}{450}$$

$$r = 3$$

The rate is 3%.

b. How do you find the interest when you know the rate and the principal?

The rate is 3% or $\frac{3}{100}$. The interest is n dollars and the ratio of the interest to the principal is $\frac{n}{450}$.

$$\frac{3}{100} = \frac{n}{450}$$

$$\frac{3 \times 450}{100} = n$$

$$13.50 = n$$

The interest is $13.50

c. How do you find the principal when you know the rate and the interest?

The rate is 3% or $\frac{3}{100}$. The interest is $13.50 and the ratio of the interest to the principal, p dollars, is $\frac{13.50}{p}$.

$$\frac{3}{100} = \frac{13.50}{p}$$

$$3p = 1350$$

$$p = 450$$

The principal is $450.

Exercises 10-5e

1. Find the interest for one year on $1,800 at 4%.
2. What is the rate if the interest for one year on $1,250 is $75?
3. What amount of interest will $900 earn in a year at $3\frac{1}{2}\%$?
4. What principal will earn $42.50 in a year at 5%?
5. Find the interest on $250 at 4% for a year.

10-6. Summary.

In order to prepare for the ideas of percent, you learned about ratio and proportion. You will use the principles of ratio and proportion in your later work in mathematics in geometry and in numerous applications to science problems as well as to problems of business.

Percent is widely used in practical affairs. Everyone must know how to deal with problems involving percent. Only selected examples of the use of percent can be included in this chapter, but any percent problem can be solved in the same way. All you need to know is how to translate the facts of the problem into a proportion. You can solve the proportion as you demonstrated by your work in this chapter.

The ratio of a number a to a number b $(b \neq 0)$ is the quotient $\frac{a}{b}$.

The Comparison Rule: If $\frac{a}{b} = \frac{c}{d}$ $(b \neq 0,\ d \neq 0)$ then $a \cdot d = b \cdot c$.

Two physical quantities are said to be proportional to one another if the ratio of their measures is always the same.

Any fraction, $\frac{a}{b}$, can be expressed as a percent by finding c so that $\frac{a}{b} = \frac{c}{100} = c \times \frac{1}{100} = c\%$.

Any fraction, $\frac{a}{b}$, can be written in decimal form and then expressed as a percent by finding c so that $\frac{a}{b} = c \times .01 = c\%$.

The sign $\approx$ means approximately equal. It is used when numbers are rounded, and when approximate rather than exact values are appropriate.

Example: if $x = \frac{2}{3}$, then $x \approx 67\%$.

A budget is a plan for spending money. The ratios of the separate parts to the total income are often expressed as percents.

A commission is the payment, often based on a percent of the amount of his sales, that is paid to a salesman for his service.

A discount is the amount subtracted from the marked price.

The sale price is the marked price less the discount.

The percent of increase or decrease is the ratio of the increase to the original amount.

Interest is a fee paid for the use of money.

The principal is the amount of money on which interest is paid. It is the amount borrowed or loaned.

The rate of interest is the ratio of the interest to the principal. The rate of interest is stated as a percent.

10-7. Chapter Review.

Exercises 10-7

1. Write the ratio of

   a. 22 to 24

   b. 63 to 56

2. Which of the following ratios are equal?

   a. $\frac{22}{11}$, $\frac{8}{4}$ c. $\frac{48}{16}$, $\frac{54}{18}$

   b. $\frac{14}{7}$, $\frac{18}{8}$ d. $\frac{18}{17}$, $\frac{72}{68}$

3. In each of the following find the number for c which will make the statement true.

   a. $\frac{20}{6} = \frac{c}{9}$ c. $\frac{3}{c} = \frac{15}{75}$

   b. $\frac{c}{90} = \frac{19}{30}$ d. $\frac{81}{100} = \frac{12}{c}$

4. A picture 4 inches wide and $4\frac{1}{2}$ inches high must be reduced to fit a 2 inch newspaper column. How high will the picture be, in the newspaper?

5. A triangle has sides of length 11 inches, 8 inches, and 6 inches. In another triangle, the measures of the sides have the same ratio. The shortest side of the second triangle is 9 inches in length. Find the length of each of the other sides.

6. Write the following as percents.

   a. .02 b. .055 c. $\frac{2}{3}$ d. $\frac{1}{8}$

7. The monthly take-home pay for the Donovan family is $400. The payment on the mortgage for their house is $80 a month. What percent of the Donovan income is needed to pay the mortgage.

8. If the 4% sales tax on a new car was $108, what was the price of the car, not including the tax?

9. The records of Central High School show that about 76% of the students can be expected to buy the yearbook. The enrollment this year is 1862. How many pupils can be expected to buy the book this year?

10. Mr. Stephens earns a commission of 40% of his sales. His sales in October amounted to $2,450. What was his commission in October?

11. Boys' sweaters marked at $12.00 were on sale at a discount of 30%. Find the sale price.

12. When calls on coin telephones increased in price from 5 cents to 10 cents, what was the percent of increase?

13. State the property of equal rational numbers which is useful in solving a proportion.

    Use $\frac{15}{27} = \frac{20}{36}$ in your statement.

14. The baseball team of Room 106 won 13 games of the 16 the boys played. The Room 107 team won 15 games of the 18 they played. Which team had the better record?

15. Find the interest on $1,050 at 3% for one year.

10-8 Cumulative Review

Exercises 10-8

HOW ARE YOU DOING?

Answer exercises 1 - 3 True or False

1. Every number can be completely factored in only one way.

2. In the base twelve system $7 \times 9 = 53_{\text{twelve}}$.

3. The numeral "4" (four) has the same meaning in the base five number system as in our base ten number system.

4. In which base has this multiplication been performed?

$$\begin{array}{r} 123 \\ 32 \\ \hline 312 \\ 1101 \\ \hline 11322 \end{array}$$

5. Write the numeral for 17 in

   a. Base five

   b. Base eight

   c. Base two

6. Use the associative property of multiplication so that the product can be found easily.

$$31 \cdot 5 \cdot 2$$

7. Write the following statements in words:

   a. $8 < 12$

   b. $34 > 32$

   c. $5 > 3 > 2$

8. State the commutative property for addition using the symbols a and b for any whole numbers.

9. The following points are suggested by objects in your kitchen. Point A is suggested by the refrigerator handle. Point B is suggested by the foot of the kitchen table. Point C is suggested by the faucet on your kitchen sink.

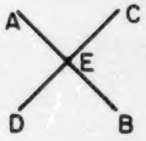
   a. How many different planes contain A and B?

   b. How many different lines contain A and C?

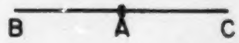
   c. How many different planes contain A, B, and C?

10. In each case, describe the union of the two sets given below:

a. The set of points on the C side of $\overleftrightarrow{AB}$ and the A side of $\overleftrightarrow{CD}$.



b. The set of points on $\overrightarrow{AB}$ and the set of points on $\overrightarrow{AC}$.



11. In each pair of rational numbers below, list the larger of the two.

a. $\frac{2}{6}$, $\frac{4}{10}$

b. 13, $\frac{67}{8}$

c. $\frac{9}{11}$, $\frac{11}{17}$

12. Perform the following operations.

a. $6.04 + 5.073 - 2.909$

b. $8.3 - 3.041 - 2.998$

13. Write the following as percents.

a. $\frac{4}{5}$

b. .026

c. $\frac{5}{6}$

d. $\frac{3}{8}$

14. The Fairlawn basketball team won 18 games of the first 25 played. The team won 5 of the next 9 games. Did the team improve its record?

INDEX

The reference is to the page on which the term occurs.

addition
    in base five, 41
    of rational numbers, 214
    rules, 226
angle, 241, 258, 259
associative property
    of addition, 78, 79, 81, 82, 104
    of multiplication, 80, 81, 82, 104
Babylonian numerals, 19
base, 30,64
base changing, 53
base five numerals, 53
    addition, 41
    division, 52
    multiplication, 48, 50
    subtraction, 46
base ten numerals, 28
bases other than 5 and 10, 56-63
betweenness, 94-96, 104, 231
binary system, 59, 64
boundary, 256
broken line curve, 254
budgets, 366, 382
closed region, 256
closure, 87, 104
commissions, 369, 382
common denominator, 201
    least common denominator, 202
common factor, 155, 171
common multiple, 165, 166, 168, 169, 170, 171
commutative property
    of addition, 73, 74, 77, 81, 104
    of multiplication, 74, 77, 80,81, 104, 157, 158
comparison rule, 289, 305
complete factorization, 155, 156, 163, 164, 168, 171
composite number, 156, 157, 170
complex fraction, 302, 305
counting number, 71, 99, 103, 147, 170
curve, 254
    broken line, 254
    simple closed, 253, 254
decimal
    addition of, 319
    decimals and percent, 361
    division of, 324
    expanded form, 327, 336
    multiplication of, 323
    notation, 311, 336
    numerals, 21
    subtraction of, 321
    system, 22, 64

digits, 22, 64
discounts, 369, 382
distributive property, 82, 83, 84, 86, 104
divisible, 158
divisibility, 160, 163, 168
  by 3, 160, 161, 162
  by 4, 5, 6, or 9, 162
division
  meaning of, 183
  of whole numbers, 175
  of rational numbers, 207
  rule, 226, 283
duodecimal system, 58, 64
Egyptian numerals, 17
elements of a set, 68
empty set, 129, 141
even number, 159
expanded form, 27, 64
  in base five, 39
  of decimals, 336
exponent, 30, 64
exterior of closed curve, 255
factor, 153, 154, 158, 170
fractions, 180
  addition, 214, 226
  division, 207
  multiplication, 193
  simplifying, 197
geometric line, 110
geometry, 107
greatest common multiple, 169
half-line, 238, 240, 259
half-planes, 238
half-spaces, 236
Hindu-Arabic system, 21
identity element
  for addition, 100, 101, 104
  for multiplication, 97, 104
improper fraction, 294, 305
interest, 379, 382
interior of closed curve, 255
inverse operations, 90, 93, 104
if-then statements, 2, 3
intersect, 111
intersections, 141
  lines and planes, 132, 135
  sets, 128, 129, 141
  two lines, 135
    Case I, Case II, Case III, 132-133
Karl Friedrich Gauss, 6
Konigsberg bridges problem, 14
largest common factor, 155
least common multiple, 165, 166, 168, 170, 171
line, 110
lines and space, 110
logical reasoning, 2, 3

marked price, 370
mixed numbers, 293, 305
multiple, 148, 153, 158, 165, 170
multiple ratios, 351
multiplicand, 153
multiplier, 153
names, 121
name of plane, 123
naming lines, 112
naming points, 108
net price, 370
non-metric geometry, 107
number line, 95, 96, 104, 267
numerals, 17, 63, 65
numerals in base five, 33
odd number, 159
one-to-one correspondence, 70, 248, 259
order, 304
parallel lines, 141
percent, 352, 361
    increase and decrease, 375, 382
perfect numbers, 165
place value, 23, 65
place value in base five, 38, 54
plane, 114, 115
point of intersection, 111
points, 107
positional form (or notation), 312, 313
power, 65
powers of ten, 31
prime factors, 155, 170
prime numbers, 149, 150, 151, 157, 170
principal, 379, 382
probability, 9, 10, 11
product rule, 196, 226
proper fraction, 294, 305
properties (meaning of), 190, 191
properties of one, 96, 97, 98, 102, 111, 153
properties of zero, 99, 100
proportion, 346
rate of interest, 370, 379, 382
ratio, 343
    definition, 344
    multiple ratios, 351
rational numbers, 177, 185
    addition of, 214
    comparison of, 285
    decimal form, 315
    division of, 207
    multiplication (by whole numbers), 203
    on the number line, 270
    operations with, 188
    subtraction of, 220
ray, 240, 258
reciprocals, 280, 305
    rule, 282

region, 256, 259
represent a plane, 122
Roman numerals, 19
rounding decimal numerals, 332
sale price, 370, 382
segment, 231, 258
separations, 236, 259
set, 67, 104, 128
sieve of Erastosthenes, 149, 157
simple closed curves, 253, 259
simplification rule, 226
skew lines, 141
space, 113
straight line, 110, 112
subtraction rules, 220, 226
summary, 139
symbols, 17, 65, 121
    > (greater than), 75, 76, 95, 104
    < (less than), 75, 76, 95, 104
    ≠ (not equal to), 75, 76, 104
    · (times), 75, 76, 104
    ≈ (approximately equal to), 362
tests for divisibility, 163
triangle, 244, 258
    angles of, 245
    sides of, 245
twin primes, 152
union of sets, 234, 258, 259
unique factorization property, 158
vertex, 242
whole number, 71, 92, 99, 104, 169, 170